PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA/ISAS

RECEIVED: *December 16, 2003*
ACCEPTED: *January 9, 2004*

Exact solution of quantum field theory on noncommutative phase spaces

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ABSTRACT: We present the exact solution of a scalar field theory defined with noncommuting position and momentum variables. The model describes charged particles in a uniform magnetic field and with an interaction defined by the Groenewold-Moyal star-product. Explicit results are presented for all Green's functions in arbitrary even spacetime dimensionality. Various scaling limits of the field theory are analysed non-perturbatively and the renormalizability of each limit examined. A supersymmetric extension of the field theory is also constructed in which the supersymmetry transformations are parametrized by differential operators in an infinite-dimensional noncommutative algebra.

KEYWORDS: Renormalization Regularization and Renormalons, Non-Commutative Geometry, Matrix Models.

In memory of Ian I. Kogan.

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1. Introduction and summary

Despite the enormous amount of activity centered around the study of noncommutative field theories (see [1]–[3] for reviews), some of the most fundamental questions surrounding them have yet to be answered to full satisfaction. Foremost among these is whether or not these models can make sense as renormalizable, interacting quantum field theories. Non-local effects such as the interlacing of ultraviolet and infrared scales [4] make conventional renormalization schemes, such as the wilsonian approach, seemingly hopeless. In fact, this mixing is known to render some noncommutative field theories non-renormalizable, even if their commutative counterparts are renormalizable. A class of such models are asymptotically-free noncommutative field theories [5, 6], wherein the combination of asymptotic freedom and UV/IR mixing forces the couplings to run to their trivial gaussian fixed points in the ultraviolet scaling limit. Such theories possess non-trivial interactions only when they are defined with a finite ultraviolet cutoff. This result is obtained by expansion about a translationally invariant vacuum of the quantum field theory.

The analysis of UV/IR mixing and the renormalization of noncommutative scalar field theory to all orders of perturbation theory has been carried out in [7]–[9]. For at least certain classes of noncommutative field theories, it may be that unusual effects such as UV/IR mixing are merely perturbative artifacts which disappear when resummed to all orders. This is of course a non-perturbative issue, which motivates the search for examples of exactly solvable noncommutative field theories. An example of such a model was presented and analysed in [10]. It describes charged scalar fields on two-dimensional Euclidean spacetime in a constant external magnetic field and with a non-local quartic interaction. It may be thought of as the natural modification of a quantum field theory on a noncommutative space to one with noncommuting momentum space coordinates as well, i.e. as a quantum field theory on a noncommutative *phase* space. These models provide concrete examples in which one can see genuine quantum field theoretic issues, such as renormalization, at work in non-trivial exactly solvable theories.

In [10] the exact two-point Green's function of this model was obtained and the non-perturbative non-renormalizability of the field theory established. In this paper we will elaborate on this analysis and present various extensions and generalizations of this class of noncommutative field theories. In particular, we will extend the solution to arbitrary even dimensionality and higher order Green's functions, as well as presenting a detailed analysis of the various field theory limits which are possible in these models. In the course of this analysis, we will encounter *new* types of complex matrix models with external fields, for which we provide the necessary technical tools for analysis. Other aspects of these types of quantum field theories are discussed in [11]–[14], while their one-particle sectors are analysed in [15]–[22]. Related exact calculations in nonrelativistic noncommutative field theory may be found in [23, 24].

1.1 Description of the model

The model we shall study in this paper is a particular deformation of scalar field theory in both position and momentum space. Consider complex scalar fields $\Phi(x)$ on even dimensional Euclidean spacetime $\mathbb{R}^{2n}$ in an external constant magnetic field B_{ij}, and with an interaction defined by replacing the usual pointwise multiplication of fields with the Groenewold-Moyal star-product, defined such that the components of the spatial coordinates $x = (x^1, \ldots, x^{2n})$ satisfy the relations

$$x^i \star x^j - x^j \star x^i = 2\,\mathrm{i}\,\theta^{ij}\,. \tag{1.1}$$

From (1.1) and the Baker-Campbell-Hausdorff formula it follows that the star-product of two plane waves is given by

$$\mathrm{e}^{\,\mathrm{i}\,k\cdot x} \star \mathrm{e}^{\,\mathrm{i}\,q\cdot x} = \mathrm{e}^{\,-\,\mathrm{i}\,k_i\,\theta^{ij}\,q_j}\ \mathrm{e}^{\,\mathrm{i}\,(k+q)\cdot x}\,. \tag{1.2}$$

Thus the interaction can be computed by expanding the fields in plane waves, i.e. by Fourier transformation, and (1.2) defines the interaction part of the action for all Schwartz functions $\Phi(x)$. The background magnetic field, on the other hand, may be realized as the curvature of Landau momentum operators $P_i = -\,\mathrm{i}\,\frac{\partial}{\partial x^i} - B_{ij}\,x^j$ with

$$[P_i, P_j] = -2\,\mathrm{i}\,B_{ij}\,. \tag{1.3}$$

The free part of the action is then defined by replacing the laplacian operator of conventional scalar field theory with the Landau hamiltonian P_i^2. The $2n \times 2n$ constant skew-symmetric matrices $\theta = (\theta^{ij})$ and $B = (B_{ij})$, characterizing respectively the noncommutativity of the coordinates and momenta, are assumed to be non-singular.

It was shown in [12] that there is a natural regularization of these models which allows, in a simple manner, to make precise mathematical sense of the functional integrals defining the quantum field theory. Rather than expanding the boson fields in plane waves, it is more natural to expand them in the eigenfunctions of the Landau hamiltonian P_i^2 and thereby diagonalize the free part of the action. The Landau eigenfunctions $\phi_{\boldsymbol{\ell},\boldsymbol{m}}(x)$ are labelled by two sets of quantum numbers $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n), \boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, and the corresponding energy eigenvalues depend on only half these quantum numbers $\boldsymbol{\ell}$. By expanding the fields as $\Phi(x) = \sum_{\boldsymbol{\ell},\boldsymbol{m}} M_{\boldsymbol{m},\boldsymbol{\ell}}\,\phi_{\boldsymbol{\ell},\boldsymbol{m}}(x)$, the formal functional integration then becomes an integration over all complex expansion coefficients $M_{\boldsymbol{m},\boldsymbol{\ell}}$. This suggests a regularization of the quantum field theory in which the collections of Landau quantum numbers $\boldsymbol{\ell}$ and $\boldsymbol{m}$ are each restricted to a finite set which is in one-to-one correspondence with the set $\{1, 2, \ldots, N\}$, where $N < \infty$. With this regularization, the functional integral becomes a finite dimensional integral over $N \times N$ complex matrices. Similar matrix approximation schemes have also been studied in [9] and [25]–[28]. We remark that these matrix models of noncommutative field theory are not the same as those based on lattice regularization [29, 30], and in particular the scaling limits required in the two approaches are very different.

As discussed in [12], in the case of complex Φ^4-theory deformed as described above, a finite matrix dimension N amounts to having both an ultraviolet and infrared cutoff at the same time, as there exist duality transformations which exchange infrared and ultraviolet divergences, while the action and all regularized Green's functions of the model are duality invariant. In this paper we will carry this approach one step further and describe precisely how to remove the regularization, i.e. pass to the field theory limit $N \to \infty$. In general, the interaction part of the action has a quite complicated form in the Landau basis (This is of course also true in the commutative case [31]). However, the quantum duality symmetry in this family of models suggests that they should be special when $B = \pm\,\theta^{-1}$ [12]. The crucial point is that at the special points $B = \pm\,\theta^{-1}$, the interactions acquire a simple form as well owing to a remarkable closure property of the Landau eigenfunctions under the star-product, and the field theory can be mapped *exactly* onto a matrix model [13]. The significance of the points $B = \pm\,\theta^{-1}$ has been previously noted within different field theoretic contexts in [30]–[33] and within the context of noncommutative quantum mechanics in [15, 16, 21].

In this paper we will concentrate for the most part on extracting Green's functions of the noncommutative field theory in the limit where the regulator is removed. The latter limit corresponds to the large N limit of the matrix model, and we can make it non-trivial simply by taking the 't Hooft limit of the matrix model [10]. We thereby determine, in a completely non-perturbative way, how to renormalize the parameters of the model, i.e. give them a particular dependence on the regulator N such that the $N \to \infty$ limit exists and is non-trivial. Even though this limit is essentially unique at the level of the matrix model, we will see that there is still some freedom left in defining the limit at the level of the field theoretic Green's functions. In [10] the two-point Green's function was derived for one such limit in which it is translationally invariant. In the following we will use duality to argue that there is another limit, which mathematically is simply obtained from that of [10] by Fourier transformation, but which has a very different physical interpretation. We will use this result to argue that there may exist limits "in between" these other two scaling limits.

1.2 Summary of main results

From the mapping sketched above onto a finite-dimensional matrix model, we will derive a number of important consequences which lead to the exact solution of the noncommutative quantum field theory. Foremost among these features is the fact that the regularized field theory, for any interaction potential, is exactly solvable in the sense that there is a closed formula for the partition function at finite N. In fact, we will establish the formal integrability of a more general matrix model which in the field theory amounts to having, in addition to the magnetic field, a confining electric field background for the charged scalars. In [13] this was established within the framework of the hamiltonian quantization of the nonrelativistic, $2n+1$-dimensional fermion version of this field theory. It was shown that this model has a dynamical symmetry group $\mathrm{GL}(\infty)\times\mathrm{GL}(\infty)$, with the free part of the model given as a linear superposition of Cartan elements of the representation and the interaction proportional to a tensor Casimir operator. The dynamical symmetry allows the explicit and straightforward construction of the eigenstates of the hamiltonian, and some remnant of it should survive in the high-temperature limit of the corresponding partition function, which is essentially what we deal with here. In the following we will explore the integrability of this system from a different point of view. We will derive a determinant formula for the matrix integral and hence show that it is a tau-function of the integrable, two-dimensional Toda lattice hierarchy. The $\mathrm{GL}(\infty)\times\mathrm{GL}(\infty)$ dynamical symmetry then presumably manifests itself as the rotational symmetry of the free fermion representation of the vertex operator construction of the Toda tau-function. The limit in which the electric potential is turned off reduces the partition function *exactly* to a KP tau-function.

We will then study the large N limit of the original matrix model. The Schwinger-Dyson equations of the matrix model succinctly sum up all divergences which one would encounter when solving the quantum field theory in perturbation theory. We shall compute the field theoretic Green's functions from the matrix model correlators, which also requires us to perform sums over Landau levels of certain combinations of the Landau eigenfunctions. The latter quantities capture the spacetime dependence of the Green's functions. As a warm-up and as a simple example where we can give complete results, we will first consider the limiting case in which the kinetic energy in the action is dropped. In this case we compute *all* Green's functions explicitly.

We will then present the detailed computation of the two-point function $G(x,y)=\langle\Phi^\dagger(x)\,\Phi(y)\rangle$ of the full model in various scaling limits of the quantum field theory with a complex Φ^4 interaction potential. In the limit $N\to\infty$ with the quantity $\Lambda^{2n}=N\,\sqrt{\det(B/4\pi)}$ held fixed, we find that the exact propagator admits the Fourier integral representation

$$G_{\rm ir}(x,y)=-\int_{|p|\le 4\,\sqrt{\pi}\,\Lambda}\frac{\mathrm{d}^{2n}p}{(2\pi)^{2n}}\ \Lambda^{-2}\,W\left(\Lambda^{-2}\ \left(p^2+\mu^2\right)\right)\ \mathrm{e}^{\,\mathrm{i}\,p\cdot(x-y)}\ , \tag{1.4}$$

where μ is the mass of the scalar field $\Phi(x)$, and $W(\lambda)$ is the propagator of the underlying matrix model which may be determined as the solution of the Schwinger-Dyson equations in the large N limit. The quantity Λ can be interpreted here as an ultraviolet cutoff, and in the limit $\Lambda\to\infty$ the function W reduces to

$$W(\lambda)=\frac{\lambda-\sqrt{\lambda^2+4\Lambda^{2n-4}\,g}}{2\Lambda^{2n-4}\,g} \tag{1.5}$$

with g the bare Φ^4 coupling constant. This result was derived in [10] in the two-dimensional case. We will then argue that all higher Green's functions can be obtained from the two-point function by using Wick's theorem, i.e. all connected Green's functions in this particular scaling limit vanish, mainly as a consequence of the usual large N factorization in the matrix model. This type of scaling seems to be generic to the matrix regularization of noncommutative field theories [26, 30]. In particular, the continuum limits of the lattice derived matrix models of noncommutative field theory seem to be most naturally associated with the trivial gaussian fixed point of the corresponding commutative theory [34, 35]. Notice, however, that if we replace the bare coupling g by the renormalized coupling constant $\Lambda^{2n-4}\,g$, then the function $\Lambda^{-2}\,W(\Lambda^{-2}\,\xi)$ has a more complicated form reflecting the interacting nature of the quantum field theory, provided we simply interpret Λ as the finite mass scale set by the magnetic field. Then there is no trouble going to the full quantum field theory limit, and this suggests that an alternative physical interpretation for the infrared limit may also be given. This interpretation is consistent with the old folklore that spacetime noncommutativity provides a means of regulating ultraviolet divergences in quantum field theory.

We then exploit the duality discussed above to show that there exists another scaling limit of the quantum field theory, leading to a Green's function which is essentially the Fourier transform of (1.4) (up to rescalings). It corresponds to taking the limit $N\to\infty$ with $\Lambda^{2n} = N^{-1}\sqrt{\det(B/4\pi)}$ fixed and it produces an ultra-local propagator

$$G_{\rm uv}(x,y) = -\delta^{(2n)}(x-y)\ \Lambda^{-2}\,W\Big((4\pi\,\Lambda)^2+\Lambda^{-2}\,\mu^2\Big)\ , \qquad |x|\leq\frac{\Lambda^{-1}}{\sqrt\pi}\ , \tag{1.6}$$

where the parameter Λ now has the interpretation of an infrared cutoff. As remarked above, however, there is an alternative interpretation of both of these limits in which Λ may be interpreted simply as the mass scale set by the background magnetic field, i.e. $\Lambda^{2n} = \sqrt{\det(B/4\pi)}$. This follows from a remarkable scale invariance of the regularized Green's functions (and the underlying matrix model) which shows that, in the regularized quantum field theory, the length scale is set by the magnetic length $l_{\rm mag} = \det(B)^{-1/4n}$. The two scaling limits thus described can be interpreted as zooming into short distances $\ll l_{\rm mag}$ in space in the infrared limit leading to (1.4), and zooming out to large distances $\gg l_{\rm mag}$ in the ultraviolet limit leading to (1.6). With this latter physical interpretation, one can proceed to analyse whether there exist well-defined intermediate limits with no zooming in or out of space at all. We thus find that the renormalization of the interacting noncommutative field theory depends crucially on the particular scaling limits that one takes, and also on their physical interpretations.

We will also describe some other extensions of the present class of models which have the potential of providing exactly solvable, renormalizable, and interacting noncommutative field theories. In particular, we present a detailed construction of a supersymmetric extension of the noncommutative scalar field theory in a background magnetic field. One of the most interesting aspects of these models that we shall find is that even at the level of the non-interacting theory, noncommutative geometry appears to play a role, in that the supersymmetry transformations are parametrized by elements of an infinite-dimensional, noncommutative algebra (or equivalently by infinite matrices). We use this observation to

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explicitly construct the most general naively renormalizable, noncommutative supersymmetric field theory, while it does not seem possible to construct local actions which possess this infinite parameter supersymmetry. This model thereby represents a supersymmetric extension of noncommutative field theory for which, like the scalar models studied in this paper, no commutative counterpart exists and noncommutativity is an intrinsic feature. We will not deal with the problem of actually solving any of these generalizations in this paper. Their exact solutions are left for future work which has the potential of deepening our insights into the basic structures underlying noncommutative quantum field theory.

1.3 Outline

In the ensuing sections we shall give detailed derivations of the results reported above, and describe many other interesting features of these noncommutative field theories. The structure of the remainder of this paper is as follows. In section 2 we give the precise definition of the quantum field theory, describe its symmetries, and briefly review the duality of [12]. In section 3 we derive in detail, following [13], the relationship between the quantum field theory and a large N matrix model, focusing our attention on the two-dimensional case for simplicity and ease of notation. We then use this mapping to describe the nonperturbative regularization that will be employed for most of this paper. In section 4 we prove that the regularized field theory is exactly solvable by deriving a closed formula for the finite N partition function, and relate it to the Toda lattice hierarchy. We also derive and explicitly solve the equations of motion in the large N limit, and then evaluate the exact vacuum amplitude of the field theory. In section 5 we study the Green's functions of the field theory in two dimensions, starting with those for which the free part of the action is dropped. All connected Green's functions are obtained in this limit and are invariant under Fourier transformation, as expected by duality. We then give a detailed derivation of the two-point function obtained in [10] in the infrared limit described above, and further argue that the connected correlation functions of the field theory are all trivial in this limit. In section 6 we derive the generalizations of the preceeding results to arbitrary even spacetime dimensionality. In section 7 the various other possible scaling limits of the quantum field theory are explored, including the infrared limit, and the different physical interpretations of the infrared and ultraviolet limits. We also briefly describe how non-trivial interactions can be obtained by (small) perturbations of the model. In section 8 we present the detailed construction of the relativistic fermion version of the scalar noncommutative field theory, and combine it with the bosonic model to construct a supersymmetric, interacting noncommutative field theory in a background magnetic field. Appendix A describes an alternative interpretation of the model as a particular generalization of mean field theory for conventional (commutative) complex Φ^4-theory, providing another motivation for the present work. In appendix B we sketch a "stringy" interpretation of the duality of the model. The remaining appendices C–G contain more technical details of the calculations.

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2. Formulation of the model

In this section we will start by defining precisely the noncommutative quantum field theory that we shall study in this paper. We shall also present a detailed analysis of the symmetries possessed by this model, and describe in what sense it may be regarded as an exactly solvable quantum field theory.

2.1 Definitions

We will consider the class of noncommutative scalar field theories defined by classical actions of the form

$$\tilde{S}_{\rm b} = \int {\rm d}^{2n}x\ \Big[\Phi^\dagger(x)\left(-\sigma\,\mathsf{D}^2 - \tilde{\sigma}\,\tilde{\mathsf{D}}^2 + \mu^2\right)\Phi(x) + V_\star\left(\Phi^\dagger \star \Phi\right)(x)\Big]\,, \tag{2.1}$$

where $\sigma + \tilde{\sigma} \geq 0$ and

$$\mathsf{D}_i = \partial_i - {\rm i}\,B_{ij}\,x^j\,, \tag{2.2}$$

$$\tilde{\mathsf{D}}_i = \partial_i + {\rm i}\,B_{ij}\,x^j \tag{2.3}$$

with $\partial_i = \partial/\partial x^i$, $i = 1,\dots,2n$. Here $\Phi(x)$ is a massive, complex scalar field of mass dimension $n-1$ coupled minimally to a constant, background magnetic field proportional to the $2n\times 2n$ real, non-degenerate antisymmetric matrix $B = (B_{ij})$, and in the background of a confining electric potential proportional to $(Bx)^2$. Throughout we work on even-dimensional Euclidean spacetime $\mathbb{R}^{2n}$, and we use the notation $\Phi^\dagger(x) \equiv \overline{\Phi(x)}$ with the bar indicating complex conjugation. The interaction term is defined by a potential of the general form

$$V(w) = \sum_{k\geq 2} \frac{g_k}{k}\,w^k\,, \tag{2.4}$$

and the subscript $\star$ means that the real scalar fields $X(x) = \Phi^\dagger\star\Phi(x)$ are multiplied together in $V_\star(X)(x)$ using the usual Groenewold-Moyal star-product, $(X)^k_\star \equiv X \star X \star \cdots \star X$ (k times), where

$$f \star f'(x) = (2\pi)^{-4n}\int {\rm d}^{2n}k \int {\rm d}^{2n}q\ \tilde{f}(k)\,\tilde{f}'(q)\ \ {\rm e}^{\,{\rm i}\,k\cdot\theta q}\ \ {\rm e}^{\,{\rm i}\,(k+q)\cdot x}\,. \tag{2.5}$$

Here $\tilde{f}$ is the Fourier transform of the field f with the convention

$$\tilde{f}(k) = \int {\rm d}^{2n}x\ \ {\rm e}^{\,-{\rm i}\,k\cdot x}\,f(x)\,, \tag{2.6}$$

and for any two momentum space vectors k and q we have defined their skew-product by

$$k\cdot\theta q = \theta^{ij}\,k_i\,q_j\,. \tag{2.7}$$

We will assume that the noncommutativity parameter matrix $\theta = (\theta^{ij})$ is non-degenerate. We also define

$$x\cdot By = B_{ij}\,x^i\,y^j\,. \tag{2.8}$$

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Most of the results we shall describe in the following will be obtained from the noncommutative complex Φ^4 field theory with interaction potential given by

$$V(w)=\frac{g}{2}\,w^2\,, \tag{2.9}$$

and in the special instances when either σ or $\tilde{\sigma}$ vanishes with $\sigma+\tilde{\sigma}=1$. These cases correspond to charged particles in a background magnetic field alone. In section 5.1 we shall consider the case where both $\sigma=\tilde{\sigma}=0$. The cases $\sigma,\tilde{\sigma}>0$ will have the effect of removing the degeneracies of the Landau levels in the magnetic field problem. The special case $\sigma=\tilde{\sigma}=\frac{1}{2}$ in (2.1) corresponds to scalar fields in a harmonic oscillator potential alone and is closest to the conventional noncommutative field theories with no background magnetic field. The addition of such a harmonic oscillator potential to standard noncommutative Φ^4-theory is in fact necessary for its renormalization to all orders in perturbation theory [9]. However, most of our analysis will focus on the instance where $\sigma=1$, $\tilde{\sigma}=0$, and the interaction potential is given by (2.9), i.e. the noncommutative field theory with action

$$S_{\rm b}=\int \mathrm{d}^{2n}x\ \left[\Phi^\dagger(x)\left(-\mathrm{D}^2+\mu^2\right)\Phi(x)+\frac{g}{2}\,\Phi^\dagger\star\Phi\star\Phi^\dagger\star\Phi(x)\right]. \tag{2.10}$$

At the quantum level, the field theory is defined by the partition function

$$Z_{\rm b}=\int \mathrm{D}\Phi\ \mathrm{D}\Phi^\dagger\ \ \mathrm{e}^{\,-\tilde{S}_{\rm b}} \tag{2.11}$$

and the non-vanishing Green's functions

$$\begin{aligned} G_{\rm b}^{(2r)}(x_1,\dots,x_r;y_1,\dots,y_r)\ &=\ \left\langle\Phi^\dagger(x_1)\,\Phi(y_1)\cdots\Phi^\dagger(x_r)\,\Phi(y_r)\right\rangle\\ &\equiv\ \frac{1}{\tilde{Z}_{\rm b}}\int \mathrm{D}\Phi\ \mathrm{D}\Phi^\dagger\ \prod_{I=1}^{r}\Phi^\dagger(x_I)\,\Phi(y_I)\ \ \mathrm{e}^{\,-\tilde{S}_{\rm b}}\,, \end{aligned} \tag{2.12}$$

with the usual, formal functional integration measure $\mathrm{D}\Phi\ \mathrm{D}\Phi^\dagger=\prod_x \mathrm{d}\,\mathrm{Re}\,\Phi(x)\ \mathrm{d}\,\mathrm{Im}\,\Phi(x)$. The connected parts of the Green's functions (2.12) may be extracted from the generating functional which is defined by the functional integral with the coupling of the scalar fields to external sources $J(x)$ and $\overline{J}(x)$,

$$Z_{\rm b}\left[J,\overline{J}\,\right]=\int \mathrm{D}\Phi\ \mathrm{D}\Phi^\dagger\ \ \mathrm{e}^{\,-\tilde{S}_{\rm b}+\int \mathrm{d}^{2n}x\ \left[\Phi^\dagger(x)\,J(x)+\overline{J}(x)\,\Phi(x)\right]}\,. \tag{2.13}$$

We then have

$$G_{\rm b\,,\,conn}^{(2r)}(x_1,\dots,x_r;y_1,\dots,y_r)=\prod_{I=1}^{r}\frac{\delta}{\delta J(x_I)}\,\frac{\delta}{\delta\overline{J}(y_I)}\,\ln\frac{Z_{\rm b}\left[J,\overline{J}\,\right]}{Z_{\rm b}}\Bigg|_{J=\overline{J}=0}\,. \tag{2.14}$$

To make the definition of the functional integral measure somewhat more precise we will expand the fields in a convenient basis of functions. Formally, one can choose any complete orthonormal set in the Hilbert space of L^2-functions on $\mathbb{R}^{2n}$. However, different choices of basis lead to different regularization prescriptions for the quantum field theory. For example, if one uses Fourier expansion of fields in a basis of plane waves $\mathrm{e}^{\,\mathrm{i}\,k\cdot x}$, then the natural regularization imposed is the restriction of momenta $k \in \mathbb{R}^{2n}$ to the annulus $\Lambda_0 \leq |k| \leq \Lambda$, where Λ_0 is interpreted as an infrared cutoff and Λ as an ultraviolet cutoff. Removing the cutoffs then amounts to taking the limits $\Lambda_0 \to 0$ and $\Lambda \to \infty$. This procedure is known to be rather difficult to carry out in noncommutative quantum field theory, and has in fact been done only at low orders in perturbation theory due to the complications set in by UV/IR mixing. In what follows we will circumvent such problems by using a different basis of functions that enables us to make sense of the quantum field theory non-perturbatively.

2.2 Spacetime symmetries

Let us consider the spacetime symmetries of the field theory defined by the action (2.1). If $n = 1$, then the theory is rotationally invariant, because a constant magnetic field is rotationally invariant in two dimensions. However, parity is broken by the magnetic field. A parity transformation amounts to changing the sign of the magnetic field, $B \to -B$, or equivalently to interchanging $\sigma \leftrightarrow \tilde{\sigma}$. What is much less obvious is that, in any dimension, the model is also translationally invariant for $\tilde{\sigma} = 0$. For this, we accompany a translation of the scalar field $\Phi(x)$ by a gauge transformation in the constant magnetic field,

$$\Phi_a(x) \equiv \mathrm{e}^{\,\mathrm{i}\, a\cdot Bx}\, \Phi(x + a)\,. \tag{2.15}$$

Note that the product on the right-hand side of (2.15) is just *ordinary* pointwise multiplication of fields. Under (2.15), the covariant derivative of Φ transforms homogeneously,

$$\mathsf{D}_i \Phi_a = (\mathsf{D}_i \Phi)_a\,. \tag{2.16}$$

An elementary calculation using (2.5) shows that the star-product also transforms in a simple way as

$$\Phi_a^\dagger \star \Phi_a(x) = \Phi^\dagger \star \Phi\, \big(x + (\mathbb{1} - \theta B)\, a\big)\,. \tag{2.17}$$

It follows that the action (2.1) with $\tilde{\sigma} = 0$ is invariant under the substitution of Φ by Φ_a. We will refer to this symmetry as "magnetic translation invariance".

If this symmetry persists at the quantum level, then the two-point Green's function $G(x, y) = \langle \Phi^\dagger(x)\, \Phi(y)\rangle$ assumes the form

$$G(x, y) = \mathrm{e}^{\,-\,\mathrm{i}\, x\cdot By}\, g(x - y) \tag{2.18}$$

with $g(x) \equiv G(x, 0)$ a function only of one variable. This implies that the two-point function takes the same form in momentum space. Computing the Fourier transform of (2.18) gives

$$\begin{aligned} \tilde{G}(p, q) &\equiv \int \mathrm{d}^{2n}x\ \mathrm{e}^{\,\mathrm{i}\, p\cdot x} \int \mathrm{d}^{2n}y\ \mathrm{e}^{\,-\,\mathrm{i}\, q\cdot y}\, G(x, y) \\ &= \det\left(2\pi B^{-1}\right)\ \mathrm{e}^{\,-\,\mathrm{i}\, q\cdot B^{-1}p}\, g\Big(B^{-1}(q - p)\Big) \ = \ \det\left(2\pi B^{-1}\right)\, G\left(B^{-1}q, B^{-1}p\right), \end{aligned} \tag{2.19}$$

which for $B = \theta^{-1}$ is a special case of the duality to be discussed in section 2.3. To interpret our later results it is important to note that the limits $B \to 0$ and $B \to \infty$ are singular. At finite B the Green's functions in position and momentum space are both translationally invariant up to a phase. However, in the limit $B \to 0$ only the translational symmetry in position space is maintained, with $G(x,y) \to g(x-y)$. In momentum space this limit is ill-defined, but a direct computation yields $\tilde{G}(p,q) = (2\pi)^{2n}\ \delta^{(2n)}(p-q)\ \tilde{g}(p)$ and the momentum space Green's function becomes ultra-local in this limit. An analogous remark applies to the limit $B \to \infty$ with the roles of G and $\tilde{G}$ interchanged.

One can now ask if the full symplectomorphism group of $\mathbb{R}^{2n}$ is a symmetry of the field theory, as it is in the case of noncommutative gauge theories [36]. For this, we use the Weyl-Wigner correspondence to associate, for each Schwartz function $f(x)$ on $\mathbb{R}^{2n} \to \mathbb{C}$, a corresponding compact Weyl operator $\hat{f}$ by

$$\hat{f} \equiv f(\hat{x}) = \int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}}\ \tilde{f}(k)\ \mathrm{e}^{\,\mathrm{i}\,k\cdot\hat{x}}\,, \tag{2.20}$$

where $\hat{x}^i$ are the hermitean generators of the Heisenberg algebra of n degrees of freedom which is defined by the commutation relations

$$\left[\hat{x}^i\,,\,\hat{x}^j\right] = 2\,\mathrm{i}\,\theta^{ij}\,. \tag{2.21}$$

We will represent this algebra on a separable Hilbert space $\mathcal{H}$. With this correspondence, we can rewrite the action (2.1) by using the usual rules

$$\widehat{f\star g} = \hat{f}\,\hat{g} \ = \ f(\hat{x})\,g(\hat{x})\,, \tag{2.22}$$

$$\widehat{\overline{f}} = \hat{f}^\dagger\,, \tag{2.23}$$

$$\widehat{\partial_i f} = \frac{1}{2\,\mathrm{i}}\ \left(\theta^{-1}\right)_{ij}\ \left[\hat{x}^j\,,\,\hat{f}\right]\,, \tag{2.24}$$

$$\int \mathrm{d}^{2n}x\ f(x) \ = \ \sqrt{\det(4\pi\theta)}\ \ \mathrm{Tr}_{\,\mathcal{H}}\,(\hat{f}\,)\,, \tag{2.25}$$

where the dagger in (2.23) denotes the Hilbert space adjoint and the trace of operators over $\mathcal{H}$ in (2.25) is defined such that

$$\mathrm{Tr}_{\,\mathcal{H}}\left(\mathrm{e}^{\,\mathrm{i}\,p\cdot\hat{x}}\right) = \frac{(2\pi)^{2n}}{\sqrt{\det(4\pi\theta)}}\ \delta^{(2n)}(p)\,. \tag{2.26}$$

By using (2.22) and (2.24), along with the star-product identity

$$x^i\star f(x) = x^i\,f(x) + \mathrm{i}\,\theta^{ij}\,\partial_j f(x)\,, \tag{2.27}$$

we may also derive the correspondence

$$\widehat{x^i\,f} = \frac{1}{2}\left(\hat{x}^i\,\hat{f} + \hat{f}\,\hat{x}^i\right). \tag{2.28}$$

It follows that the actions of the covariant derivatives in (2.1) on the Hilbert space $\mathcal{H}$ are given by

$$\begin{aligned}\widehat{\mathsf{D}_i f} &= -\frac{\mathrm{i}}{2}\left(\left(\theta^{-1}\right)_{ij} - B_{ij}\right)\hat{x}^j\,\hat{f} + \frac{\mathrm{i}}{2}\left(\left(\theta^{-1}\right)_{ij} + B_{ij}\right)\hat{f}\,\hat{x}^j\,,\\ \widehat{\tilde{\mathsf{D}}_i f} &= -\frac{\mathrm{i}}{2}\left(\left(\theta^{-1}\right)_{ij} + B_{ij}\right)\hat{x}^j\,\hat{f} + \frac{\mathrm{i}}{2}\left(\left(\theta^{-1}\right)_{ij} - B_{ij}\right)\hat{f}\,\hat{x}^j\,.\end{aligned} \tag{2.29}$$

We see that the points where $B\,\theta = \pm\,\mathbb{1}$ have the special property that only one of the terms on the right-hand sides of the equalities in (2.29) is non-zero, and the thus the action assumes a particularly simple form. In particular, at the special point in parameter space where

$$B\,\theta = \mathbb{1}\,, \tag{2.30}$$

the action (2.1) is mapped under the above correspondence to

$$\begin{aligned}\tilde{S}_{\rm b} \;=\; \sqrt{\det(4\pi\theta)}\;\; {\rm Tr}_{\;\mathcal{H}}\Big[\Phi^\dagger(\hat{x})\Big(\tilde{\sigma}\,(B\hat{x})^2+\mu^2\Big)\Phi(\hat{x})+\sigma\,\Phi^\dagger(\hat{x})\Phi(\hat{x})(B\hat{x})^2 \,+ \\ +\,V\Big(\Phi^\dagger(\hat{x})\Phi(\hat{x})\Big)\Big]\,,\end{aligned} \tag{2.31}$$

and an analogous formula in the case $B\,\theta = -\mathbb{1}$ with the interchange $\sigma \leftrightarrow \tilde{\sigma}$. At these points, and only at these points, the action depends solely on the combinations $\Phi^\dagger(\hat{x})\Phi(\hat{x})$ and $\Phi(\hat{x})\Phi^\dagger(\hat{x})$. As we will see, this property makes the field theory exactly solvable at the full quantum level. Note that the representations of the two covariant derivatives in (2.29) also commute at these special points, so that the Hilbert space $\mathcal{H}$ provides a Morita equivalence bimodule for the corresponding Heisenberg algebras that they generate. This equivalence is described from another point of view in appendix B.

The action (2.31) for $\tilde{\sigma} = 0$ possesses a large symmetry

$$\Phi(\hat{x}) \;\longmapsto\; U(\hat{x})\,\Phi(\hat{x})\,, \qquad \Phi^\dagger(\hat{x}) \;\longmapsto\; \Phi^\dagger(\hat{x})\,U(\hat{x})^{-1}\,, \tag{2.32}$$

where $U(\hat{x})$ is any invertible operator on $\mathcal{H}$. In the spacetime picture, this ${\rm GL}(\infty)$ symmetry corresponds to the complex, fundamental star-gauge transformations

$$\Phi(x) \;\longmapsto\; U \star \Phi(x)\,, \qquad \Phi^\dagger(x) \;\longmapsto\; \Phi(x) \star U^{\star-1}(x)\,, \tag{2.33}$$

where $U(x)$ is a star-invertible field on $\mathbb{R}^{2n}$,

$$U \star U^{\star-1}(x) = U^{\star-1} \star U(x) = 1\,. \tag{2.34}$$

With both $\sigma = \tilde{\sigma} = 0$ the action has an even larger ${\rm GL}(\infty)\times{\rm GL}(\infty)$ symmetry

$$\Phi(\hat{x}) \;\longmapsto\; U(\hat{x})\,\Phi(\hat{x})\,V(\hat{x})^{-1}\,, \qquad \Phi^\dagger(\hat{x}) \;\longmapsto\; V(\hat{x})\,\Phi^\dagger(\hat{x})\,U(\hat{x})^{-1}\,. \tag{2.35}$$

As we will see, this ${\rm GL}(\infty)$ symmetry drastically simplifies the calculation of the Green's functions of the model. The unitary subgroup of this huge symmetry corresponds to invariance of the model under canonical transformations of $\mathbb{R}^{2n}$ [36]. The symplectomorphism $W_{1+\infty}$-symmetry at the critical point (2.30) has also been observed within a different context in the corresponding noncommutative quantum mechanics [20, 22].

A somewhat different interpretation of the special point (2.30) may also be given. The actions of the covariant momentum operators (2.2) and (2.3) may be written, by using (2.27), as

$$\begin{aligned}\mathsf{D}_i\Phi(x) &= (\mathbb{1}+B\,\theta)_i{}^j\,\Big(\partial_j\Phi(x)+{\rm i}\,B'_{jk}\,x^k\star\Phi(x)\Big)\,,\\ \tilde{\mathsf{D}}_i\Phi(x) &= (\mathbb{1}-B\,\theta)_i{}^j\,\Big(\partial_j\Phi(x)+{\rm i}\,B''_{jk}\,x^k\star\Phi(x)\Big)\,,\end{aligned} \tag{2.36}$$

where

$$B' = \frac{1}{\mathbb{1} + B\,\theta}\,B\,, \qquad B'' = \frac{1}{\mathbb{1} - B\,\theta}\,B\,. \tag{2.37}$$

This relationship is a very simple version of the Seiberg-Witten map between commutative and noncommutative descriptions of the same theory [32]. One can either work with the new background magnetic fields B' and B'' using explicit noncommutative star-products in the definition of the Landau momentum operators, or else one can work with the original magnetic field B using the ordinary commutative product. The two descriptions are the same up to an overall rescaling. However, one of the transformations in (2.37) is always undefined at the points $B = \pm\,\theta^{-1}$, and in this sense they can be interpreted as points of "maximal" noncommutativity where the quantum field theory becomes exactly solvable.[1]

2.3 UV/IR duality

As demonstrated in [12], the noncommutative field theory in the case of a quartic interaction also possesses a novel duality symmetry. As this duality will be important for the physical interpretations of the scaling limits of the regularized field theory that we will obtain, we shall now summarize the duality transformation rules, refering to [12] for the technical details. The action (2.10) has a duality under Fourier transformation, i.e. it retains the same form when written in position or momentum space. Explicitly, it is invariant under the duality transformation

$$\begin{aligned} \Phi(x) &\longmapsto \sqrt{\left|\det(B/2\pi)\right|}\;\tilde{\Phi}(B\,x)\,, \\ \theta &\longmapsto -B^{-1}\,\theta^{-1}\,B^{-1}\,, \\ g &\longmapsto \left|\det(B\,\theta)\right|^{-1}\,g\,, \end{aligned} \tag{2.38}$$

with all other parameters of the model unchanged. This symmetry extends to the quantum level as a symmetry of rewriting the Green's functions (2.12) in momentum space,

$$\tilde{G}_{\rm b}^{(r)}(p_1,\dots,p_r;q_1,\dots,q_r) = \left|\det\left(2\pi B^{-1}\right)\right|^r\,G_{\rm b}^{(r)}(x_1,\dots,x_r;y_1,\dots,y_r) \tag{2.39}$$

with $x_I = B^{-1}\,p_I$, $y_I = B^{-1}\,q_I$, $I = 1,\dots,r$, and the transformations (2.38) implicitly understood on the right-hand side of (2.39). The key to the proof of this fact is the existence of a regularization of the quantum field theory which respects the duality. This is essentially the regularization that will be exploited in the subsequent sections. Note that at the special points $B = \pm\,\theta^{-1}$, the field theory is essentially "self-dual" in that it is completely invariant under Fourier transformation, and the duality identifies the two special points. This is consistent with what was found in the previous subsection. In appendix B we point out an alternative "stringy" interpretation of this duality at the special point.

[1] Note that the transformations (2.37) can be inverted to give $B = B'(\mathbb{1} - \theta\,B')^{-1}$ and $B = B''(\mathbb{1} + \theta\,B'')^{-1}$, in which the roles of the two special points $B' = +\theta^{-1}$ and $B'' = -\theta^{-1}$ are interchanged. The Seiberg-Witten transformation then asserts that the two points are equivalent, in that they lead to the same noncommutative quantum field theory.

3. Matrix model representation

We are interested in expressing the action (2.1) with respect to the expansion of functions on $\mathbb{R}^{2n}$ in a suitable basis. To simplify the presentation of our results we will first focus on the two-dimensional case $n = 1$. The generalizations of our results to arbitrary even dimensionality is remarkably simple and will be described later on in section 6. In this section, and until section 6, we define $\theta = \theta^{12}$ and $B = B_{12}$, both of which are assumed to be positive parameters. We will begin by deriving the matrix form of the action (2.1) for $n = 1$, and then use this representation to define the formal functional integral of the quantum field theory as an integral over infinite-dimensional matrices. This will then enable us to describe a natural non-perturbative regularization of the model.

3.1 Mapping to a matrix model

Let us begin by introducing the complex coordinates

$$z = x^1 + \mathrm{i}\,x^2\,, \qquad \overline{z} = x^1 - \mathrm{i}\,x^2\,, \tag{3.1}$$

and the ladder operators

$$\begin{aligned} \mathsf{a} &= \frac{1}{2}\left(\sqrt{\theta}\,\partial + \frac{\overline{z}}{\sqrt{\theta}}\right), & \mathsf{a}^\dagger &= \frac{1}{2}\left(-\sqrt{\theta}\,\overline{\partial} + \frac{z}{\sqrt{\theta}}\right), \\ \mathsf{b} &= \frac{1}{2}\left(\sqrt{\theta}\,\overline{\partial} + \frac{z}{\sqrt{\theta}}\right), & \mathsf{b}^\dagger &= \frac{1}{2}\left(-\sqrt{\theta}\,\partial + \frac{\overline{z}}{\sqrt{\theta}}\right). \end{aligned} \tag{3.2}$$

Here $\partial = \partial_1 - \mathrm{i}\,\partial_2$ and $\overline{\partial} = \partial_1 + \mathrm{i}\,\partial_2$. The operators (3.2) generate two commuting copies of the usual harmonic oscillator commutation relations

$$\begin{aligned} \left[\mathsf{a}\,,\,\mathsf{a}^\dagger\right] &= \left[\mathsf{b}\,,\,\mathsf{b}^\dagger\right] = 1\,, \\ \left[\mathsf{a}\,,\,\mathsf{b}\right] &= \left[\mathsf{a}\,,\,\mathsf{b}^\dagger\right] = 0\,. \end{aligned} \tag{3.3}$$

The Landau states are defined by

$$|\ell, m\rangle = \frac{(\hat{\mathsf{a}}^\dagger)^{\ell-1}}{\sqrt{(\ell-1)!}}\,\frac{(\hat{\mathsf{b}}^\dagger)^{m-1}}{\sqrt{(m-1)!}}|\mathrm{vac}\rangle\,, \tag{3.4}$$

where $|\mathrm{vac}\rangle$ is the two-particle Fock vacuum with $\hat{\mathsf{a}}|\mathrm{vac}\rangle = \hat{\mathsf{b}}|\mathrm{vac}\rangle = 0$, and ℓ, m are positive integers. The normalized Landau wavefunctions in position space are given by

$$\phi_{\ell,m}(x) = \langle x|\ell, m\rangle\,. \tag{3.5}$$

The ground state wavefunction $\phi_0(x) \equiv \phi_{1,1}(x)$ can be computed explicitly by solving the differential equations $\mathsf{a}\,\phi_0 = 0$ and $\mathsf{b}\,\phi_0 = 0$ using (3.2) to get

$$\phi_0(x) = \frac{1}{\sqrt{\pi\theta}}\;\mathrm{e}^{\,-|z|^2/2\theta}\,, \tag{3.6}$$

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while in general they can be computed by introducing the generating function

$$\mathcal{P}_{s,t}(x) = \sum_{\ell,m=1}^{\infty} \frac{s^{\ell-1}}{\sqrt{(\ell-1)!}} \, \frac{t^{m-1}}{\sqrt{(m-1)!}} \, \phi_{\ell,m}(x) \,. \tag{3.7}$$

Using (3.2)–(3.6), an elementary calculation gives (see for example [13], appendix A)

$$\mathcal{P}_{s,t}(x) = \frac{1}{\sqrt{\pi\theta}} \ \mathrm{e}^{\,-st} \ \mathrm{e}^{\,(sz+t\,\overline{z})/\sqrt{\theta}} \ \mathrm{e}^{\,-|z|^2/2\theta} \,, \tag{3.8}$$

and from (3.8) the Landau wavefunctions may be extracted as

$$\phi_{\ell,m}(x) = \frac{1}{\sqrt{(\ell-1)!\,(m-1)!}} \, \frac{\partial^{\ell-1}}{\partial s^{\ell-1}} \, \frac{\partial^{m-1}}{\partial t^{m-1}} \mathcal{P}_{s,t}(x) \bigg|_{s=t=0} \,. \tag{3.9}$$

The advantage of working with the functions $\phi_{\ell,m}(x)$ is two-fold. First of all, using the generating function (3.8) it is easy to check that they form a complete orthonormal basis of one-particle wavefunctions, and

$$\int \mathrm{d}^2x \ \overline{\phi_{\ell,m}} \star \phi_{\ell',m'}(x) = \delta_{\ell\ell'} \, \delta_{mm'} \tag{3.10}$$

with $\overline{\phi_{\ell,m}} = \phi_{m,\ell}$. Secondly, a straightforward computation using (3.8) and the representation (2.5) yields the identity (see for example [13], appendix A)

$$\mathcal{P}_{s,t} \star \mathcal{P}_{s',t'} = \frac{1}{\sqrt{4\pi\theta}} \ \mathrm{e}^{\,s't} \ \mathcal{P}_{s,t'} \,, \tag{3.11}$$

which is equivalent to the star-product projector relation

$$\phi_{\ell,m} \star \phi_{\ell',m'} = \frac{1}{\sqrt{4\pi\theta}} \, \delta_{m\ell'} \ \phi_{\ell,m'} \,. \tag{3.12}$$

This relation of course just reflects the well-known fact that the functions $\sqrt{4\pi\theta}\,\phi_{\ell,m}$ provide the Wigner representation of the rank 1 Fock space operators $\hat{\phi}_{\ell,m} = |\ell\rangle\langle m|$.

These facts suggest an expansion of the scalar fields in the action (2.1) with respect to the Landau basis as

$$\Phi(x) = \sqrt{4\pi\theta} \ \sum_{\ell,m=1}^{\infty} M_{\ell m} \ \overline{\phi_{\ell,m}}(x) \,, \qquad \Phi^\dagger(x) = \sqrt{4\pi\theta} \ \sum_{\ell,m=1}^{\infty} \overline{M_{\ell m}} \ \phi_{\ell,m}(x) \,. \tag{3.13}$$

It is natural to assemble the dimensionless complex numbers $M_{\ell m}$ into an infinite, square complex matrix $M = (M_{\ell m})_{\ell,m\geq 1}$. To ensure convergence of the expansion (3.13), we will regard M as a compact operator acting on the separable Hilbert space $\mathcal{H} = \mathcal{S}(\mathbb{N})$ of Schwartz sequences $(a_m)_{m\geq 1}$ with sufficiently rapid decrease as $m \to \infty$. Then, the projector relation (3.12) implies that the star-product of fields $\Phi(x)$ and $\Phi'(x)$ corresponds to ordinary matrix multiplication [25],

$$\Phi \star \Phi'(x) = \sqrt{4\pi\theta} \ \sum_{\ell,m=1}^{\infty} (M \cdot M')_{\ell m} \ \overline{\phi_{\ell,m}}(x) \,, \tag{3.14}$$

while (3.12) along with the orthogonality relation (3.10) implies that spacetime averages of fields correspond to traces over the Hilbert space $\mathcal{H}$,

$$\int \mathrm{d}^2x \ \Phi(x) = 4\pi\theta \sum_{m=1}^{\infty} M_{mm} \equiv 4\pi\theta \ \mathrm{Tr}_{\ \mathcal{H}}(M)\,. \tag{3.15}$$

The star-product powers $(\Phi^\dagger{\star}\Phi)_\star^k$ in the interaction potential $V_\star(\Phi^\dagger{\star}\Phi)(x)$ may thereby be succinctly written as traces of the matrices $(M^\dagger M)^k$. For the kinetic energy terms in (2.1), we use the definitions (2.2) and (3.2) to write

$$\mathsf{D}^2 = -\frac{1}{\theta}\left[(1+B\theta)^2\left(\mathsf{a}^\dagger\mathsf{a}+\frac{1}{2}\right)+(1-B\theta)^2\left(\mathsf{b}^\dagger\mathsf{b}+\frac{1}{2}\right)+\left(B^2\theta^2-1\right)\left(\mathsf{a}^\dagger\mathsf{b}^\dagger+\mathsf{a}\mathsf{b}\right)\right], \tag{3.16}$$

along with the usual harmonic oscillator creation and annihilation relations applied to the Landau wavefunctions (3.5) to get

$$\begin{aligned} \mathsf{a}\,\phi_{\ell,m} &= \sqrt{\ell-1}\ \phi_{\ell-1,m}\,, & \mathsf{a}^\dagger\,\phi_{\ell,m} &= \sqrt{\ell}\ \phi_{\ell+1,m}\,,\\ \mathsf{b}\,\phi_{\ell,m} &= \sqrt{m-1}\ \phi_{\ell,m-1}\,, & \mathsf{b}^\dagger\,\phi_{\ell,m} &= \sqrt{m}\ \phi_{\ell,m+1}\,. \end{aligned} \tag{3.17}$$

The operator $\tilde{\mathsf{D}}^2$ is obtained from (3.16) by a reflection of the magnetic field $B\to -B$.

We thereby find that the total action (2.1) may be expressed as the infinite-dimensional complex matrix model action

$$\begin{aligned} \tilde{S}_{\rm b} = {} & (\sigma+\tilde\sigma)\left(B^2\theta^2-1\right)\ \mathrm{Tr}_{\ \mathcal{H}}\left(\Gamma_\infty^\dagger\,M^\dagger\,\Gamma_\infty\,M+M^\dagger\,\Gamma_\infty^\dagger\,M\,\Gamma_\infty\right)+\\ &+\left(\sigma+\tilde\sigma+(\sigma-\tilde\sigma)B^2\theta^2\right)\ \mathrm{Tr}_{\ \mathcal{H}}\left(M\,\mathcal{E}\,M^\dagger\right)+\\ &+\left(\sigma+\tilde\sigma-(\sigma-\tilde\sigma)B^2\theta^2\right)\ \mathrm{Tr}_{\ \mathcal{H}}\left(M^\dagger\,\mathcal{E}\,M\right)+\\ &+4\pi\theta\,\mu^2\ \mathrm{Tr}_{\ \mathcal{H}}\left(M^\dagger M\right)+4\pi\theta\ \mathrm{Tr}_{\ \mathcal{H}}V\left(M^\dagger M\right), \end{aligned} \tag{3.18}$$

where we have introduced the diagonal matrix $\mathcal{E}=(\mathcal{E}_{\ell m})$ with

$$\mathcal{E}_{\ell m} = 4\pi\left(\ell-\frac{1}{2}\right)\ \delta_{\ell m} \tag{3.19}$$

and the infinite shift matrix $\Gamma_\infty=(\Gamma_{\infty,\ell m})$ with

$$\Gamma_{\infty,\ell m} = \sqrt{4\pi(m-1)}\ \delta_{\ell,m-1}\,. \tag{3.20}$$

Note that the noncommutativity parameter θ now only appears as an explicit coupling parameter in the action (3.18). The noncommutativity of spacetime has simply become the noncommutativity of matrix multiplication. The first three traces in the action (3.18) are similar to the clock and hopping terms that appear as the kinetic parts in the lattice derived matrix models of noncommutative scalar field theory [30].

The maximal symmetry group of area-preserving diffeomorphisms, which appears at the special points in parameter space where either $\sigma=0$ or $\tilde\sigma=0$, and $B\theta=\pm 1$, is particularly transparent within this formalism. At these points, the first term in (3.18) vanishes and the resulting action depends only on the combinations $M^\dagger M$ and $MM^\dagger$. In particular, for $\tilde{\sigma} = 0$ it possesses the GL(∞) symmetry

$$M \longmapsto U \cdot M \,, \qquad M^\dagger \longmapsto M^\dagger \cdot U^{-1} \,. \tag{3.21}$$

As we will discuss below, in this case the noncommutative quantum field theory has the same fixed point as the complex one-matrix model in the 't Hooft limit. Note also that in this case the Landau states diagonalize the Landau hamiltonian operator D^2, with the standard harmonic oscillator spectrum

$$-\mathsf{D}^2 \phi_{\ell,m} = 4B\left(\ell - \frac{1}{2}\right)\phi_{\ell,m} \,. \tag{3.22}$$

The GL(∞) symmetry can thereby be physically understood as a consequence of the infinite degeneracies of the Landau levels ℓ, and it acts through rotations of the magnetic quantum numbers m. The phase space now becomes degenerate and the Landau wavefunctions depend on only half of the position space coordinates, leading to a reduction of the quantum Hilbert space at $B = \theta^{-1}$. Henceforth we shall deal only with the quantum field theory defined at this special point. Note that from (3.16) it follows that while the D^2 operator at $B\theta = 1$ is given by the $\mathsf{a}, \mathsf{a}^\dagger$ oscillators, the hamiltonian $\tilde{\mathsf{D}}^2$ is given in terms of the commutant $\mathsf{b}, \mathsf{b}^\dagger$ oscillators and the eigenvalue equation (3.22) is modified to

$$-\tilde{\mathsf{D}}^2 \phi_{\ell,m} = 4B\left(m - \frac{1}{2}\right)\phi_{\ell,m} \,. \tag{3.23}$$

3.2 Regularization

Let us consider now the quantum field theory in the matrix model representation (3.18), whereby the scalar field variables can be identified with infinite matrices. The jacobian of this transformation is formally 1, and the partition function (2.11) is then given by the functional integral for the $N = \infty$ complex matrix model. With this rewriting we can now properly define the functional integration measure appearing in (2.11). Namely, we replace the infinite matrix variables by finite $N \times N$ matrices, and then take the large N limit of the matrix model. In other words, we restrict the quantum numbers of the Landau wavefunctions to $\ell, m = 1, \dots, N$ with $N < \infty$. There are several advantages to this matrix regularization. For example, as discussed in [12], a finite matrix rank N provides both a short distance and low momentum cutoff simultaneously, and thereby avoids the mixing of ultraviolet and infrared divergences which seem to make the analysis of noncommutative perturbation theory to all orders hopeless. Introducing an ultraviolet cutoff alone in the matrix form is not enough to regulate all possible divergences. Ultraviolet and infrared divergences cannot be clearly separated and one needs to regulate them both at the same time. This point will be further elucidated in section 7. In this respect, the Landau basis for the expansion of noncommutative fields is far superior to the conventional Fourier basis, in that it naturally leads to a non-perturbative regularization which avoids dealing with the usual difficulties of noncommutative quantum field theory. It is important to note though that this matrix regularization does not impose any *a priori* restrictions on the allowed noncommutativity parameters θ, in contrast to what occurs in lattice regularization [30].

However, while the matrix regularization is *close* to a cutoff in the kinetic energy term of the action (2.1), N cannot itself be an ultraviolet cutoff, because it is not determined by any mass scales of the theory. Without a dimensionful scale there can be no dimensional transmutation of the coupling constants of the model, which is the essence of renormalization in quantum field theory. The renormalization ensures finiteness of physical quantities and requires some set of observables to define renormalization conditions. The way to define the theory is explained in [26] and consists of first introducing the kinetic energy cutoff, then taking the limit $\theta \to \infty$ with the cutoff scaled such that it remains finite, and finally removing the cutoff. This limit is equivalent to the conventional large N limit of the complex matrix model [37, 38]. In other words, we should take the limit $N \to \infty$ while keeping fixed the dimensionful ratio

$$\Lambda^2 = \frac{N}{4\pi\theta}\,, \tag{3.24}$$

which simply defines the large θ limit of the model (2.1). The true ultraviolet cutoff (3.24) has a very natural physical interpretation. At $B = \theta^{-1}$, it is associated with the energy $2B(2N-1) = 16\pi\Lambda^2 - 2B$ of the $N^{\rm th}$ Landau level, which remains finite in the large N limit just described. Since $B \to 0$ as $N \to \infty$, the spacing between Landau levels also vanishes. Thus taking the limit described above is equivalent to filling the finite energy interval $[0, 16\pi\Lambda^2]$ with infinitely many Landau levels and an infinite density of states. This limit can be thereby regarded as a combination of those found in the corresponding noncommutative quantum mechanics in [15, 20, 22], whereby the Laughlin theory of the lowest Landau level is recovered, and in [16], where all Landau levels survive.

The large θ limit of the model (2.1) thereby defines a field theory with a finite cutoff. The quantum field theory in this limit coincides with the 't Hooft limit of the matrix model and, as we will see below, is exactly solvable (at the special point where $B\theta = 1$). At this stage, we should renormalize the regulated quantum field theory, i.e. fine-tune the parameters of the model in such a way that the limit $\Lambda \to \infty$ is well-defined and non-trivial. For this, we rescale the massive coupling constants μ^2 and g_k to the dimensionless ones

$$\tilde\mu^2 = \frac{\mu^2}{\Lambda^2}\,, \qquad \tilde g_k = \frac{g_k}{\Lambda^2}\,. \tag{3.25}$$

Let us now specialize to the case $\sigma = 1$, $\tilde\sigma = 0$ in (2.10). The functional integral (2.11) may then be properly defined through the large N limit of the matrix integral

$$\begin{aligned} Z_N(E) &= \int {\rm D}M~{\rm D}M^\dagger~~{\rm e}^{\,-N~{\rm Tr}\,\left(M\,E\,M^\dagger + \tilde V(M^\dagger M)\right)}\,, \\ {\rm D}M~{\rm D}M^\dagger &\equiv \prod_{\ell,m=1}^N \frac{{\rm d}M_{\ell m}~{\rm d}\overline{M_{\ell m}}}{{\rm i}\,\pi}\,, \end{aligned} \tag{3.26}$$

with the integration extending over the finite-dimensional linear space of $N \times N$ complex matrices $M = (M_{\ell m})_{1 \le \ell, m \le N}$, and

$$\tilde V(w) = \sum_{k \ge 2} \frac{\tilde g_k}{k}\, w^k \tag{3.27}$$

the renormalized interaction potential. Expectation values in the matrix model (3.26) are defined in the usual way as

$$\left\langle \mathcal{O}\left(M, M^\dagger\right)\right\rangle_E = \frac{1}{Z_N(E)} \int \mathrm{D}M ~ \mathrm{D}M^\dagger ~ \mathrm{e}^{-N ~ \mathrm{Tr}\left(M\, E\, M^\dagger + \tilde{V}(M^\dagger M)\right)} ~ \mathcal{O}\left(M, M^\dagger\right) , \tag{3.28}$$

such that they are well-defined and finite in the limit $N \to \infty$. For later convenience, we have introduced in (3.26) a more general complex matrix model depending on a generic $N \times N$, hermitean external field E, which after calculation should be set equal to the diagonal matrix

$$\tilde{\mathcal{E}}_{\ell m} = \left(\frac{16\pi\,(\ell - 1/2)}{N} + \tilde{\mu}^2\right) \delta_{\ell m} \tag{3.29}$$

appropriate to the original field theory, with $1 \leq \ell, m \leq N$. This provides a concrete, non-perturbative definition of the noncommutative quantum field theory. Of course afterwards the limit in which the ultraviolet cutoff (3.24) is removed should also be taken.

In what follows we will not be very precise about the convergence of the functional integrals (3.26) and (3.28). In the regularized theory, the convergence of the traces is obvious. This is very similar to the situation in conventional quantum field theory, whereby one usually has an action functional which is only well-defined on fields which are sufficiently smooth or of Schwartz-type, while at the quantum level the functional integral also takes into account less well-behaved functions. At the algebraic level, the large N limit should be rigorously defined by using the appropriate analog of that described for the noncommutative torus in [39].

While the above prescription is not the only possible definition of the quantum theory, we will see that it leads to a physically sensible and well-defined formulation of the quantum field theory, which is moreover exactly solvable. For this, it is crucial that an overall factor of N has scaled out in front of the action in (3.26). With this convention, the correct scaling of the free energy which is finite in the limit $N \to \infty$ is given by

$$\Xi_N(E) = -\frac{1}{N^2} \ln Z_N(E) . \tag{3.30}$$

From a field theoretical point of view, this follows from the fact that the free energy should be proportional to the number of one-particle degrees of freedom. In a free scalar quantum field theory, this would be given by Tr $\ln(-\partial^2 + \mu^2) \propto \Lambda^2\,\mathcal{V}$, where $\mathcal{V}$ is the spatial area and Λ^2 the area in momentum space. By using (3.24), this quantity is proportional to N^2 in the large N limit. We will examine the issue of other possible scaling definitions of the matrix model representation in section 7.

It is important to note here that, unlike the conventional large θ scalar field theories studied in [26] where derivative terms in the action contribute only at leading order in $\frac{1}{N}$, the model we solve here retains its kinetic energy terms exactly at $N = \infty$. The exact solvability of the model is not ruined in this case because of the persistent GL(∞) symmetry, contrary to the cases of [26] where the kinetic energy breaks this symmetry. In fact, already at the classical level one can see some important differences. The classical solutions of the $\theta = \infty$ models studied in [26] are given by hermitean matrices X obeying the matrix equation $\tilde{V}'(X) = 0$. They are of the form

$$X = \sum_\ell \mu_\ell \, \hat{\phi}_{\ell,\ell} \,, \tag{3.31}$$

where μ_ℓ are the distinct real critical points of the potential $\tilde{V}(w)$. The corresponding field configurations correspond to the standard $\theta = \infty$ GMS solitons [40] and possess an infinite degeneracy under $U(\infty)$ rotations of the matrices X, leading to a non-trivial soliton moduli space which is an infinite-dimensional grassmannian [41]. On the other hand, in the present case the classical equations of motion are

$$\tilde{\mathcal{E}} + \tilde{V}'(X) = 0 \,, \tag{3.32}$$

where $X = M^\dagger M$. The degeneracy of the soliton solutions (and the existence of the ensuing moduli space) is now lifted. The solutions of (3.32) are still of the form (3.31), but now the real numbers μ_ℓ satisfy the equation $\tilde{\mathcal{E}}_{\ell\ell} + \tilde{V}'(\mu_\ell) = 0$. For any polynomial potential V, this latter equation at large ℓ generically has only a single solution, since $\tilde{\mathcal{E}}_{\ell\ell} \to \infty$ as $\ell \to \infty$ and $\tilde{V}'(w) \simeq \tilde{g}_{\deg V} \, w^{\deg V - 1}$ for $|w| \to \infty$. Because of the kinetic energy term there exist only finitely many real soliton solutions. As we will see in the following, this fact also makes a big difference at the quantum level, wherein the saddle-point solutions may be regarded as perturbations around the GMS soliton solutions [26], and the functional integral as an integral over the soliton moduli space. Indeed, the expansion of functions on $\mathbb{R}^2$ with respect to the Landau basis may be thought of as a "soliton" expansion of fields on the noncommutative plane in terms of projection operators and partial isometries of the Heisenberg algebra.

As we will demonstrate in the next section, the crucial simplification which occurs at the special point $B = \theta^{-1}$ is that the partition function and observables of the matrix model depend only on the eigenvalues of the external field E. This enables an explicit solution of the matrix model. In section 5 we will then see that the computations of the Green's functions (2.12) boil down to extracting the matrix model solution, and sums over Landau levels of certain combinations of the Landau wavefunctions which give the spacetime dependence of the correlation functions. This will then enable a non-perturbative analysis of the possible scaling regimes in which the cutoff (3.24) can be removed.

4. Exact solution

We will now show how to solve the quantum field theory at the special point in parameter space where $B\theta = 1$. At this point we can use the correspondence with the large N limit of the complex one-matrix model to obtain an exact, non-perturbative solution of the noncommutative field theory. We will begin by showing how to solve the general external field problem defined by (3.26) at finite N, thereby establishing the formal integrability of the model. Then we shall demonstrate how to obtain the solution of (3.26) in the large N limit, and from this extract the solution of the noncommutative field theory defined by the action (2.10).

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4.1 Integrability

We will first derive a very simple determinant formula for the partition function (3.26) at finite N. While this expression will not be particularly useful for extracting the solution of the corresponding continuum noncommutative field theory, it will formally prove that the model is exactly solvable and provide insights into the potential extensions to more general exactly solvable noncommutative quantum field theories. For this, we will consider the more general class of noncommutative field theories with actions (2.1). The regulated field theory is obtained through a modification of the matrix model (3.26) defined by the partition function

$$Z_N\left(E,\tilde{E}\right)=\int \mathrm{D}M~\mathrm{D}M^\dagger~\mathrm{e}^{-N~\mathrm{Tr}\left(M\,E\,M^\dagger+M^\dagger\,\tilde{E}\,M+\tilde{V}(M^\dagger M)\right)}\,, \tag{4.1}$$

where the external fields E and $\tilde{E}$ have respective eigenvalues λ_ℓ and $\tilde{\lambda}_m$, $\ell,m=1,\dots,N$. The additional external field $\tilde{E}$ explicitly breaks the $U(N)$ symmetry of the original model. Remarkably, this extended matrix model is nevertheless still exactly solvable. For this, we use the polar coordinate decomposition of a generic $N\times N$ complex matrix M which enables us to write

$$M=U^\dagger\,\mathrm{diag}(r_1,\dots,r_N)~V\,, \tag{4.2}$$

where U and V are $N\times N$ unitary matrices, and $r_\ell\in[0,\infty)$. This transformation introduces a jacobian

$$\mathrm{D}M~\mathrm{D}M^\dagger=[\mathrm{d}U]~[\mathrm{d}V]~\prod_{\ell=1}^N\mathrm{d}h_\ell~\Delta_N[h]^2\,, \tag{4.3}$$

where $[\mathrm{d}U]$ denotes the invariant Haar measure on the unitary group $U(N)$, $h_\ell=r_\ell^2$, and

$$\Delta_N[h]=\prod_{1\le\ell<m\le N}(h_\ell-h_m) \tag{4.4}$$

is the Vandermonde determinant. Here and in the following we will not write irrelevant numerical constants (depending only on N) explicitly. Upon substituting (4.2) and (4.3) into (4.1), we observe that the integrations over $U,V\in U(N)$ decouple over the two types of external field terms in the integrand. With $H=\mathrm{diag}(h_1,\dots,h_N)$, the integration over the angular degree of freedom U may be done by using the Harish-Chandra-Itzykson-Zuber formula [42, 43]

$$\int_{U(N)}[\mathrm{d}U]~\mathrm{e}^{-N~\mathrm{Tr}\,(E\,U^\dagger\,H\,U)}=\frac{1}{\Delta_N[\lambda]\,\Delta_N[h]}~\det_{1\le\ell,m\le N}\left(\mathrm{e}^{-N\lambda_\ell h_m}\right). \tag{4.5}$$

An analogous formula holds for the integration over V with $\tilde{E}$ and $\tilde{\lambda}_m$. By expanding the determinants into sums over permutations of N objects and using the antisymmetry of the Vandermonde determinants (4.4), the partition function (4.1) may thereby be written as the eigenvalue model

$$Z_N\left(E\,,\tilde{E}\right)=\sum_{\pi\in S_N}\frac{(-1)^{|\pi|}}{\Delta_N[\lambda]\,\Delta_N[\tilde{\lambda}]}\,\prod_{\ell=1}^N\int_0^\infty \mathrm{d}h_\ell~~\mathrm{e}^{\,-N\left[\tilde{V}(h_\ell)+h_\ell(\lambda_\ell+\tilde{\lambda}_{\pi(\ell)})\right]}\,. \tag{4.6}$$

The eigenvalue integral (4.6) illustrates the necessity of the scaling $\theta\sim N$ in (3.24) in the large N limit. It is the only scaling that correctly takes into account the classical and quantum fluctuation terms in (4.6), such that the effective action for the N eigenvalues h_ℓ is of order N^2. It may be expressed in the compact form

$$Z_N\left(E\,,\tilde{E}\right)=\frac{1}{\Delta_N[\lambda]\,\Delta_N[\tilde{\lambda}]}\,\det_{1\leq\ell,m\leq N}\left(f\left(\lambda_\ell+\tilde{\lambda}_m\right)\right)\,, \tag{4.7}$$

where we have introduced the special function

$$f(\lambda)=\int_0^\infty \mathrm{d}h~~\mathrm{e}^{\,-N\left(\tilde{V}(h)+\lambda\,h\right)}\,. \tag{4.8}$$

Note that here V can be any function such that the integral in (4.8) makes sense.

One way to understand the exact solvability of this model is to note that the partition function (4.7) provides a non-trivial solution of the two-dimensional Toda lattice hierarchy [44].[2] For any polynomial potential V, we may write the determinant formula (4.7) formally as

$$Z_N\left(E\,,\tilde{E}\right)=\det_{0\leq\ell,m\leq N-1}\left(\oint_{z=0}\frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z}~z^m\oint_{\tilde{z}=0}\frac{\mathrm{d}\tilde{z}}{2\pi\,\mathrm{i}\,\tilde{z}}\tilde{z}^\ell\,f\left(z^{-1}+\tilde{z}^{-1}\right)\prod_{k=1}^\infty\,\mathrm{e}^{\,N(t_k\,z^k+\tilde{t}_k\,\tilde{z}^k)}\right)\,, \tag{4.9}$$

where we have defined the time variables

$$t_k=\frac{1}{Nk}\,\sum_{\ell=1}^N\lambda_\ell^k\,,\quad \tilde{t}_k=\frac{1}{Nk}\,\sum_{m=1}^N\tilde{\lambda}_m^k \tag{4.10}$$

with $k\geq1$. This formula is derived by using the Cauchy identity to get

$$\prod_{k=1}^\infty\,\mathrm{e}^{\,N(t_k\,z^k+\tilde{t}_k\,\tilde{z}^k)}=\prod_{\ell=1}^N\frac{1}{(1-z\,\lambda_\ell)(1-\tilde{z}\,\tilde{\lambda}_\ell)} \tag{4.11}$$

and expanding the integration contours in (4.9) to catch the residues of the poles at $z=1/\lambda_\ell$ and $\tilde{z}=1/\tilde{\lambda}_m$. This makes $Z_N(E,\tilde{E})$ a tau-function $\tau[t,\tilde{t}\,]$ of the integrable two-dimensional Toda lattice hierarchy. Its large N limit may thereby be determined from the equations of the dispersionless Toda hierarchy [45], although we shall not pursue this interesting line of approach to solving the noncommutative field theory (2.1) in this paper. A more transparent indication of the exact solvability of the matrix model (4.1) is described in appendix C.

[2]The analysis in the remainder of this subsection is geared at integrability features. It is not required in the remainder of this paper and may be skipped. The important result which we will need is the derivation of (4.25). An alternative derivation of this equation is given in appendix C.

Let us now go back to the original model (3.26), which is the $\tilde{E}=0$ limit of (4.1) and hence should correspond to a particular reduction of the Toda lattice hierarchy above. In this limit, the expression (4.7) for the matrix integral is indeterminate. One can take care of this by regularizing the $\tilde{E}\to 0$ limit in any way that removes the degeneracy, and applying l'Hôpital's rule to (4.7). In this way we get

$$Z_N(E)=\frac{1}{\Delta_N[\lambda]}\,\det_{1\le\ell,m\le N}\left(f^{(\ell-1)}(\lambda_m)\right), \tag{4.12}$$

where

$$f^{(m)}(\lambda)=\frac{1}{(-N)^m}\,\frac{\partial^m f(\lambda)}{\partial\lambda^m}=\int_0^\infty \mathrm{d}h\;h^m\;\mathrm{e}^{\,-N\left(\tilde{V}(h)+\lambda\,h\right)}\,. \tag{4.13}$$

Thus once the function (4.8) is known explicitly, then so is the partition function (3.26) for all N. The determinant formula (4.12) implies that the partition function $Z_N(E)$ is a tau-function $\tau[t]$ for the integrable KP hierarchy [46] in the times t_k of (4.10) with one-particle wavefunctions $f^{(0)}(\lambda^{-1})$, $f^{(1)}(\lambda^{-1})$, $f^{(2)}(\lambda^{-1})$, ... This follows from the fact that, for any polynomial potential V, the functions (4.13) have the correct asymptotics

$$N^{\ell-1}\,f^{(\ell-1)}\left(\lambda^{-1}\right)=\lambda^{\ell-1}\left(1+O\left(\lambda^{-1}\right)\right)\qquad\text{as}\qquad\lambda\to\infty \tag{4.14}$$

such that (4.12) is a KP tau-function. Note while generic external field matrix models have other properties and correspond to particular reductions of the KP hierarchy [46], the partition function (3.26) is *exactly* a KP tau-function. Its $W_{1+\infty}$ symmetry then manifests itself as the symmetry of the noncommutative quantum field theory under area-preserving diffeomorphisms which was described in section 2.2.

We can also see this connection at the level of the equations of motion. The Schwinger-Dyson equations for the matrix model (3.26) follow from the identities

$$\int \mathrm{D}M\;\mathrm{D}M^\dagger\;\sum_{\ell=1}^N\frac{\partial}{\partial M_{\ell m}}\left(M_{\ell m'}\;\mathrm{e}^{\,-N\,\mathrm{Tr}\left(M\,E\,M^\dagger+\tilde{V}(M^\dagger M)\right)}\right)=0\,, \tag{4.15}$$

which give rise to a system of differential equations for the partition function (4.12),

$$\frac{1}{N}\sum_{\ell=1}^N\frac{\partial}{\partial E_{m\ell}}\left[\tilde{V}'\left(-\frac{1}{N}\,\frac{\partial}{\partial E}\right)_{\ell m'}+E_{m'\ell}\right]Z_N(E)=0\,. \tag{4.16}$$

Both the integration measure and action in (3.26) are invariant under arbitrary unitary transformations of E. The partition function $Z_N(E)=Z[\lambda]$ thereby depends only on the N eigenvalues λ_ℓ of the external field E and is symmetric under permutation of them (This is manifest in the determinant form (4.12)). Thus only N of the N^2 Schwinger-Dyson equations (4.16) are independent. Writing the diagonalization explicitly as

$$E=\mathcal{U}^\dagger\;\mathrm{diag}(\lambda_1,\dots,\lambda_N)\;\mathcal{U} \tag{4.17}$$

with $\mathcal{U}$ unitary, by using the chain rule the derivatives appearing in (4.16) may be rewritten via second order perturbation theory according to

$$\frac{\partial}{\partial E_{mm'}}=\sum_{\ell=1}^{N}\mathcal{U}_{m'\ell}^{\dagger}\,\mathcal{U}_{\ell m}\,\frac{\partial}{\partial\lambda_\ell}+\sum_{\ell'\neq\ell}\frac{\mathcal{U}_{\ell' m}\,\mathcal{U}_{m'\ell}^{\dagger}}{\lambda_{\ell'}-\lambda_\ell}\,\sum_{k=1}^{N}\mathcal{U}_{\ell k}^{\dagger}\,\frac{\partial}{\partial\mathcal{U}_{\ell' k}}\,. \tag{4.18}$$

Let us now specialize to our main model of interest, the noncommutative complex Φ^4 field theory defined by the potential (2.9). In this case (4.8) is a transcendental error function that can be written in the form [47]

$$f(\lambda)=\frac{1}{\zeta+\sqrt{\zeta^2+\alpha(\zeta)}}\,,\qquad\zeta=\lambda\,\sqrt{\frac{N}{2\tilde{g}}}\,, \tag{4.19}$$

where $\alpha(\zeta)$ is a monotonic function that increases from $\alpha(0)=\frac{4}{\pi}$ to $\alpha(\infty)=2$. It is important to note though that the integrability property holds for any potential V. The exact solvability of the noncommutative field theory is a consequence of its underlying matrix representation, rather than of a special choice of interaction.

In this case, written in terms of the eigenvalues, the N independent equations of motion read

$$\left[-\frac{\tilde{g}}{N^2}\,\frac{\partial^2}{\partial\lambda_m^2}-\frac{\tilde{g}}{N^2}\,\sum_{\ell\neq m}\frac{1}{\lambda_\ell-\lambda_m}\left(\frac{\partial}{\partial\lambda_\ell}-\frac{\partial}{\partial\lambda_m}\right)+\frac{1}{N}\,\lambda_m\,\frac{\partial}{\partial\lambda_m}+1\right]Z[\lambda]=0\,. \tag{4.20}$$

Contracting (4.20) with $\lambda_m^{-k'}$ and collecting coefficients of the terms λ_m^{-k}/N^2 leads after some algebra to a set of differential constraint equations in terms of the time variables t_k of (4.10) of the form

$$\sum_{k=2}^{\infty}t_{k+k'-2}\ \mathsf{L}_k[t]Z[\lambda]=0\,, \tag{4.21}$$

where

$$\mathsf{L}_k[t]=\tilde{g}\,\sum_{n=1}^{k-1}\frac{\partial}{\partial t_n}\,\frac{\partial}{\partial t_{k-n}}+\sum_{n=1}^{\infty}n\,t_n\,\frac{\partial}{\partial t_n}+\frac{\partial}{\partial t_k}+\frac{\partial}{\partial t_{k-2}}-2\,\delta_{k,2}\,. \tag{4.22}$$

These equations coincide with the Virasoro constraints of the KP hierarchy [46]. However, while all of this provides a nice characterization of the exact solvability of the underlying matrix model, it is not very convenient for dealing explicitly with its large N limit, which is what is needed for the precise connection with the original noncommutative field theory. The way to tackle the $N\to\infty$ limit is the subject of the next subsection.

4.2 The master equation

To deal with the solution of (4.20) in the large N limit, we define the function

$$W(\lambda_m)=\frac{1}{N}\,\frac{\partial\ln Z[\lambda]}{\partial\lambda_m}\,, \tag{4.23}$$

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and introduce the spectral density of eigenvalues of the external matrix E,

$$\rho(\lambda) = \frac{1}{N}\ \mathrm{Tr}\ \delta(\lambda - E) = \frac{1}{N}\sum_{\ell=1}^{N} \delta(\lambda - \lambda_\ell)\,. \tag{4.24}$$

At $N = \infty$, the eigenvalues λ_ℓ are assumed to be of order 1, and (4.24) becomes a continuous function of λ with support on some finite interval $[a_1, a_2]$ on the real line. Setting $\lambda_m = \lambda$ in (4.23), a simple power counting argument shows that the derivative $\partial W/\partial\lambda$ which arises in the first term of (4.20) is of order $\frac{1}{N}$ and so can be dropped at $N = \infty$. Then, the differential equations (4.20) become

$$-\tilde{g}\, W^2(\lambda) - \tilde{g}\int_{a_1}^{a_2} \mathrm{d}\lambda'\ \rho(\lambda')\ \frac{W(\lambda') - W(\lambda)}{\lambda' - \lambda} + \lambda\, W(\lambda) + 1 = 0\,, \qquad \lambda \in [a_1, a_2]\,. \tag{4.25}$$

It is straightforward to obtain a perturbative solution to the non-linear integral equation (4.25) by writing a power series expansion for the function (4.23),

$$W(\lambda;\tilde{g}) = \sum_{k=0}^{\infty} \tilde{g}^k\ W^{(k)}(\lambda)\,. \tag{4.26}$$

By substituting this expansion into (4.25), we can then compute the iterative solution

$$\begin{aligned} W^{(0)}(\lambda) &= -\frac{1}{\lambda}\,, \\ W^{(k)}(\lambda) &= \sum_{l=0}^{k-1} \frac{W^{(l)}(\lambda)\, W^{(k-l-1)}(\lambda)}{\lambda} + \\ &\quad + \int_{a_1}^{a_2} \frac{\mathrm{d}\lambda'}{\lambda}\ \rho(\lambda')\ \frac{W^{(k-1)}(\lambda') - W^{(k-1)}(\lambda)}{\lambda' - \lambda} \end{aligned} \tag{4.27}$$

with $\lambda \in [a_1, a_2]$ and $k \geq 1$. The free solution $W^{(0)}(\lambda)$ of course follows directly from the gaussian , $\tilde{g} = 0$ partition function

$$Z_N^{(0)}(E) = \mathrm{e}^{\,-N\ \mathrm{Tr}\ \ln E}\,. \tag{4.28}$$

An alternative interpretation of the function (4.23) as the resolvent of a related hermitean one-matrix model is given in appendix C.

The exact, non-perturbative solution of (4.25) is worked out in appendix D. For the logarithmic derivative of the partition function, we find

$$\begin{aligned} W(\lambda_m) &= \frac{\lambda_m}{2\tilde{g}} - \frac{\sqrt{(\lambda_m - b_1)(\lambda_m - b_2)}}{2\tilde{g}} - \\ &\quad - \frac{1}{2N}\sum_{\ell=1}^{N} \frac{\sqrt{(\lambda_\ell - b_1)(\lambda_\ell - b_2)} - \sqrt{(\lambda_m - b_1)(\lambda_m - b_2)}}{(\lambda_m - \lambda_\ell)\,\sqrt{(\lambda_\ell - b_1)(\lambda_\ell - b_2)}}\,. \end{aligned} \tag{4.29}$$

The parameters b_1 and b_2 of this rational solution are determined by the non-linear constraints

$$b_1 + b_2 = -\frac{2\tilde{g}}{N}\sum_{\ell=1}^{N} \frac{1}{\sqrt{(\lambda_\ell - b_1)(\lambda_\ell - b_2)}}\,,$$

$$3\left(b_1^2+b_2^2\right)+2\,b_1b_2+8\tilde{g} = -\frac{8\tilde{g}}{N}\sum_{\ell=1}^{N}\frac{\lambda_\ell}{\sqrt{(\lambda_\ell-b_1)(\lambda_\ell-b_2)}}\,. \tag{4.30}$$

4.3 The exact vacuum amplitude

The solution of the previous subsection represents the large N limit of the general external field matrix model (3.26). For the noncommutative field theory of interest, we should now set the external field E equal to (3.29). For this, we shift the eigenvalues $\lambda_m \mapsto \lambda_m - \tilde{\mu}^2$ in (4.29), (4.30), and set

$$\lambda_m = 16\pi\,\frac{m}{N}\,, \tag{4.31}$$

where we ignore the zero-point energy shift which vanishes in the large N limit. We then need the function $W(\lambda)$ in the limit $N\to\infty$, $m\to\infty$ with $\frac{m}{N}\in[0,1]$ fixed. The spectral density

$$\rho(\lambda) = \frac{1}{16\pi} \tag{4.32}$$

is flat, and it is supported on $\lambda\in[0,16\pi]$. The resulting integrals appearing in (4.29,4.30) can be done explicitly, and after some algebra we arrive finally at

$$\begin{aligned} W_{\rm b}(\lambda) &= \frac{\lambda+\tilde{\mu}^2}{2\tilde{g}} - \frac{\sqrt{(\lambda-b_1)(\lambda-b_2)}}{2\tilde{g}} + \\ &\quad + \frac{1}{16\pi}\,\ln\left|\frac{\sqrt{b_1(\lambda-b_2)}-\sqrt{b_2(\lambda-b_1)}}{\sqrt{(b_1-16\pi)(\lambda-b_2)}-\sqrt{(b_2-16\pi)(\lambda-b_1)}}\right|\,, \end{aligned} \tag{4.33}$$

where the parameters b_1 and b_2 are determined through

$$\begin{aligned} b_1+b_2+2\tilde{\mu}^2 &= \frac{\tilde{g}}{4\pi}\,\ln\left|\frac{\sqrt{b_1}-\sqrt{b_2}}{\sqrt{b_1-16\pi}-\sqrt{b_2-16\pi}}\right|\,, \\ (b_1+b_2)^2 &= \frac{\tilde{g}}{2\pi}\left(\sqrt{b_1b_2}-\sqrt{(b_1-16\pi)(b_2-16\pi)}\right) - 3\tilde{\mu}^4 - 8\tilde{g}\,. \end{aligned} \tag{4.34}$$

The free energy $-\ln Z_{\rm b}$ of the noncommutative quantum field theory, representing the exact (connected) vacuum amplitude, can now be obtained by integrating the function (4.33) over $\lambda\in[0,16\pi]$. Using the permutation symmetry of the finite N partition function $Z[\lambda]$ in the eigenvalues λ_ℓ and the boundary conditions (4.34), we arrive after some algebra at

$$\begin{aligned} \ln Z_{\rm b} &= \lim_{N\to\infty}\frac{\ln Z_N}{N^2} \\ &= \frac{1}{32\pi}\left[\left(\sqrt{b_1b_2}+\sqrt{(b_1-16\pi)(b_2-16\pi)}\right)\times\right. \\ &\qquad \times\ln 4\left(\sqrt{b_1}-\sqrt{b_2}\right)\left(\sqrt{b_1-16\pi}-\sqrt{b_2-16\pi}\right) - \end{aligned}$$

$$- \sqrt{b_1 b_2}\, \ln b_1 b_2 - \sqrt{(b_1 - 16\pi)(b_2 - 16\pi)}\, \ln(b_1 - 16\pi)(b_2 - 16\pi) \Big] +$$

$$+ \frac{2\pi}{\tilde{g}} \left(4\tilde{\mu}^2 + 64 + b_1 + b_2 - 2\sqrt{(b_1 - 16\pi)(b_2 - 16\pi)} \right) -$$

$$- \frac{\pi}{4\tilde{g}^2} \Big[(b_1 + b_2)\Big((b_1 + b_2)^2 - 3\tilde{\mu}^4 \Big) + (b_1 - b_2)^2 \Big(2(b_1 + b_2) + 4\tilde{\mu}^2 \Big) \Big] . \tag{4.35}$$

The loop amplitude (4.33) is a multi-valued analytic function on the complex λ-plane with a square root branch cut along the interval $[b_1, b_2]$. From the constraint equations (4.34), it is straightforward to see that the two branch points b_1 and b_2 are always complex-valued. Moreover, the one-cut solution (4.33) is an analytic function of the parameters $\tilde{\mu}^2$ and $\tilde{g}$. Thus the matrix model exhibits *no* non-trivial critical behaviour as the coupling constant $\tilde{g}$ is varied, and there is no scaling limit at large N with an approach to a phase transition. This is completely consistent with the fact that the regulated noncommutative field theory already describes a *continuum* model. The problem of removing the ultraviolet regularization (3.24) will be analysed in the next section.

5. Correlation functions

In this section we shall investigate the spacetime dependence of the Green's functions (2.12) using the exact, non-perturbative solution of the matrix model that we obtained in the previous section. As an illustration of the power of working in the Landau basis for the expansion of noncommutative fields, and also as a warmup to some of the general effects that we will see, we shall first investigate a novel property of these correlators in a particular truncation of the field theory which has an extended $\mathrm{GL}(\infty) \times \mathrm{GL}(\infty)$ symmetry. Then we will proceed to obtain *exact* expressions, valid to all loop orders, for the connected Green's functions of the noncommutative quantum field theory. This will enable us to describe explicitly the non-perturbative scaling limits of the model.

5.1 Green's functions in static and free limits

An important feature in the analysis of the Green's functions (2.12) is that we use *ordinary* products of the fields. A natural question which then arises is as to what the difference is between these functions and those defined using star-products, as may seem more appropriate to a noncommutative field theory with a star-unitary symmetry. To get a flavour for the answer to this question, we will find an L^2-integration kernel $\mathcal{G}_L(x_1, \dots, x_L)$ with the property that

$$\int \mathrm{d}^2 x\ f_1 \star \cdots \star f_L(x) = \int \mathrm{d}^2 x_1\ \cdots \int \mathrm{d}^2 x_L\ \mathcal{G}_L(x_1, \dots, x_L)\ f_1(x_1) \cdots f_L(x_L) \tag{5.1}$$

for any collection of L fields $f_1, \dots, f_L$. For $L = 2$ this kernel was computed in [12]. The identity (5.1) thereby produces a convolution formula

$$\prod_{I=1}^{r} \int \mathrm{d}^2 x_I \int \mathrm{d}^2 y_I\ \mathcal{G}_{2r}(x_1, y_1, \dots, x_r, y_r)\ G^{(r)}_{\mathrm{b}}(x_1, \dots, x_r; y_1, \dots, y_r) =$$

$$= \int \mathrm{d}^2 x\ \left\langle \left(\Phi^\dagger \star \Phi \right)^r_\star (x) \right\rangle \tag{5.2}$$

for the integrated Green's functions of the noncommutative quantum field theory in terms of correlators of composite operators.

The kernel $\mathcal{G}_L$ bears a remarkable relation to the matrix model constructed in this paper. Defining $\delta_y(x) = \delta^{(2)}(x-y)$, from the definition (5.1) it follows that

$$\mathcal{G}_L(x_1,\dots,x_L) = \int \mathrm{d}^2x \ \delta_{x_1} \star \cdots \star \delta_{x_L}(x)\,. \tag{5.3}$$

Using completeness of the Landau wavefunctions on $\mathbb{R}^2$ to write

$$\delta_y(x) = \sum_{\ell,m=1}^{\infty} \phi_{\ell,m}(x)\,\phi_{m,\ell}(y) \tag{5.4}$$

and the projector relations (3.12), we may compute the star-products appearing in (5.3) according to

$$\delta_{x_1} \star \delta_{x_2}(x) = \frac{1}{\sqrt{4\pi\theta}} \sum_{\ell,m=1}^{\infty} \phi_{\ell,m}(x) \sum_{k=1}^{\infty} \phi_{\ell,k}(x_2)\,\phi_{k,m}(x_1)\,. \tag{5.5}$$

By doing this an additional $L-2$ times in (5.3), and then integrating over x using the orthonormality relation (3.10), we arrive at a trace formula for products of the Landau wavefunctions,

$$\begin{aligned} \mathcal{G}_L(x_1,\dots,x_L) &= \left(\frac{1}{\sqrt{4\pi\theta}}\right)^{L-2} \sum_{m_1,\dots,m_L=1}^{\infty} \phi_{m_1,m_2}(x_L)\,\phi_{m_2,m_3}(x_{L-1})\cdots\phi_{m_L,m_1}(x_1) \\ &\equiv \left(\frac{1}{\sqrt{4\pi\theta}}\right)^{L-2} \mathrm{Tr}_{\,\mathcal{H}}\Big(\phi(x_L)\,\phi(x_{L-1})\cdots\phi(x_1)\Big)\,. \end{aligned} \tag{5.6}$$

The physical significance of this formula is clear. Before regularization and at $B=\theta^{-1}$, the $\theta\to\infty$ limit of the matrix model action (3.18) is an invariant function of the combination $M^\dagger M$ and thereby possesses an extended $\mathrm{GL}(\infty)\times\mathrm{GL}(\infty)$ symmetry $M \mapsto U\cdot M\cdot V^{-1}$, $M^\dagger \mapsto V\cdot M^\dagger\cdot U^{-1}$. In this limit, the kinetic energy completely drops out and the field theory possesses only static configurations. Because of this huge symmetry, the only non-vanishing correlators of the large N complex matrix model are those which depend solely on traces of powers $(M^\dagger M)^r$. Using the field expansions (3.13), it follows then that the right-hand side of (5.6) gives the exact spacetime dependence of an L-point Green's function (2.12) in this limit, with $L=2r$. In other words, the solution to the star-product relation (5.1) is determined by the *bare* Green's functions in the limits of interest here, obtained essentially by dropping the kinetic energy terms in (2.1), i.e. $\sigma=\tilde\sigma=0$.

The integration kernel (5.6) is computed explicitly for all L in appendix E. For the cases $L=2r$ which are relevant for the complex scalar field theory, we have

$$\begin{aligned} \mathcal{G}_{2r}(x_1,\dots,x_{2r}) &= \left(\frac{1}{2\pi\theta}\right)^{2(r-1)} \delta^{(2)}\left(\sum_{I=1}^{2r}(-1)^I\,x_I\right)\times \\ &\qquad \times \exp\left(\frac{\mathrm{i}}{r}\sum_{I<J}(-1)^{J-I}\,(J-I-r)\,x_I\cdot Bx_J\right)\,. \end{aligned} \tag{5.7}$$

Note that this result holds for any interaction potential V. The full Green's functions (2.12) in the given limit are obtained by taking the sum of (5.7) over all permutations of the coordinates $x_1, \ldots, x_{2r}$,[3] and then multiplying it by matrix model correlators of $\mathrm{Tr}\,(M^\dagger M)^r$ which carry the detailed information about the potential (2.4) of the quantum field theory. These matrix averages are well-known to be calculable through a generating function which is defined by the contour integration [48]

$$\frac{1}{2} \oint_{\substack{[-\alpha,\alpha] \\ w\neq z}} \frac{\mathrm{d}w}{4\pi\,\mathrm{i}} \, \frac{w\,\tilde{V}'(w)}{z^2-w^2} \sqrt{\frac{z^2-\alpha^2}{w^2-\alpha^2}} = \frac{1}{N} \sum_{r=0}^{\infty} \frac{1}{z^{2r+1}} \left\langle \mathrm{Tr}\, \left(M^\dagger M\right)^r \right\rangle_{E=0} , \tag{5.8}$$

where the normalized expectation values on the right-hand side of (5.8) are defined by (3.28) in the large N limit, and the parameter α is determined in terms of the rescaled coupling constants of the potential (3.27) as the real positive solution of the equation[4]

$$\frac{\tilde{\mu}^2}{2}\,\alpha^2 + \sum_{k\geq 2} \frac{(2k-1)!}{k\,\Big[(k-1)!\Big]^2\,4^k}\;\tilde{g}_k\;\alpha^{2k} = 1\,. \tag{5.9}$$

The connected part of the $2r$-point Green's function is of order $1/N^{r-1}$ in the large N limit, due to the usual factorization of correlators in the matrix model at $N=\infty$. We will elucidate this point further in section 5.3. Notice also that the large θ bare propagator

$$\mathcal{G}_2(x,y) = \delta^{(2)}(x-y) \tag{5.10}$$

is ultra-local due to the persistent $\mathrm{GL}(\infty)\times\mathrm{GL}(\infty)$ symmetry.[5] This is no longer true, however, for the higher order Green's functions, which respect the magnetic translation symmetry described in section 2.2.

Now let us examine the opposite extreme where the interaction potential is turned off, and compare these exact results with the *free* propagator

$$C_\mu(x,y) = \langle x|\frac{1}{\hat{\mathsf{D}}^2+\mu^2}|y\rangle \tag{5.11}$$

[3]Note that (5.7) is not real-valued, because even for real functions f and g, $\overline{f\star g} = g\star f \neq f\star g$ in general. It becomes real after symmetrization over the spacetime coordinates.

[4]This result is derived by introducing the resolvent function $R(z) = \frac{1}{N}\left\langle \mathrm{Tr}\;\frac{z}{z^2-M^\dagger M}\right\rangle_{E=0}$, whose large z expansion coincides with the right-hand side of (5.8). One can then write an appropriate Schwinger-Dyson equation analogous to (4.15) for this correlator, which at $N=\infty$ takes the form [48]

$$\oint_{\mathcal{C}} \frac{\mathrm{d}w}{4\pi\,\mathrm{i}}\;\frac{w\,\tilde{V}'(w)}{z^2-w^2}\;R(w) = R^2(z) + O\left(\frac{1}{N^2}\right),$$

where the contour $\mathcal{C}$ encloses the singularities of $R(z)$, but not the point $w=z$, with counterclockwise orientation in the complex z-plane. Assuming that the only singularity of $R(z)$ is a branch cut across a single connected interval $[-\alpha,\alpha]\subset\mathbb{R}$, the solution of this equation coincides with the left-hand side of (5.8). Then (5.9) follows by imposing the asymptotic boundary condition $R(z)\simeq\frac{1}{z}+O(z^{-2})$ as $z\to\infty$ on this solution.

[5]The form (5.10) follows immediately from (5.6) by completeness of the Landau wavefunctions, or alternatively from the definition (5.1) by the well-known star-product identity $\int \mathrm{d}^2x\; f\star f'(x) = \int \mathrm{d}^2x\; f(x)\,f'(x)$.

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of a charged scalar particle of mass μ in the constant electromagnetic background. The computation of (5.11) is presented in appendix F and one finds

$$C_\mu(x,y) = \frac{1}{4\pi}\ \mathrm{e}^{\,-|x-y|^2/2\theta - \mathrm{i}\,x\cdot By} \int_0^\infty \mathrm{d}u\ \frac{\mathrm{e}^{\,-u}}{\sqrt{u^2+|x-y|^2\,u/\theta}}\ \left(\frac{u}{u+|x-y|^2/\theta}\right)^{\mu^2\,\theta/4} . \tag{5.12}$$

In particular, by setting $\mu^2=0$ in (5.12), the integral can be done explicitly and yields the free massless propagator

$$C_0(x,y) = \frac{1}{4\pi}\ \mathrm{e}^{\,-\mathrm{i}\,x\cdot By}\ K_0\left(\frac{|x-y|^2}{2\theta}\right) , \tag{5.13}$$

where K_0 is the modified Bessel function of the second kind of order 0. From (5.12) we see that the ultraviolet problem is unchanged by the presence of the electromagnetic field $B=1/\theta$, i.e. the singularity at $x=y$ is the same as that for $B=0$, while from (5.13) we see that the magnetic field acts like a mass and cures the infrared problem in the massless case. In particular, as shown in appendix F, the $\theta\to\infty$ limit of the free propagator yields an ultra-local spacetime dependence *only* for massive fields $\mu^2>0$, while the result (5.10) is valid for all values of the coupling constants of the field theory. In the massless case (5.13), the standard logarithmic ultraviolet behaviour appears only at very short length scales $|x-y|\ll\sqrt{\theta}$. As we will see more of in the following, these properties are the remnants of the UV/IR mixing phenomenon within the present class of noncommutative quantum field theories.

5.2 The exact propagator

In this subsection we will compute the exact two-point function of the quantum field theory, which by using the field expansions (3.13) can be written formally as

$$G(x,y) = \left\langle \Phi^\dagger(x)\,\Phi(y)\right\rangle = 4\pi\theta \sum_{\ell,m,\ell',m'=1}^\infty \left\langle M^\dagger_{m\ell}\,M_{\ell' m'}\right\rangle_{E=\tilde{\mathcal{E}}}\ \phi_{\ell,m}(x)\,\phi_{m',\ell'}(y)\ . \tag{5.14}$$

To make sense of (5.14), we restrict the sums to $\ell,\ell'<N$ (the sums over m and m' need no regularization) and compute the matrix correlator in the 't Hooft limit of the matrix model. The action and measure in the functional integral (3.28) are invariant under unitary transformations (3.21) with $U^\dagger=U^{-1}$. We can thereby explicitly make this transformation in the matrix integral and then integrate over the unitary group. At finite N the integral we need is [49]

$$\int_{U(N)}[\mathrm{d}U]\ U^\dagger_{k\ell}\,U_{\ell' k'} = \frac{1}{N}\ \delta_{kk'}\ \delta_{\ell\ell'} , \tag{5.15}$$

which can be derived by using the invariance properties of the Haar measure. As a consequence, the matrix expectation values appearing in (5.14) are given by

$$\left\langle M^\dagger_{m\ell}\,M_{\ell' m'}\right\rangle_{E=\tilde{\mathcal{E}}} = \frac{1}{N}\,\delta_{\ell\ell'}\left\langle\left(M^\dagger M\right)_{mm'}\right\rangle_{E=\tilde{\mathcal{E}}} = -\frac{1}{N^2}\,\delta_{\ell\ell'}\,\frac{\partial\ln Z_N(E)}{\partial E_{m'm}}\Bigg|_{E=\tilde{\mathcal{E}}} . \tag{5.16}$$

Using (4.23) and (4.31), the Green's function (5.14) is therefore given by the large N limit of

$$G_N(x,y) = -\frac{1}{N} \sum_{\ell=1}^{N} W(\lambda_\ell)\, \gamma_\ell(x,y)\,, \tag{5.17}$$

where we have introduced the sum over Landau levels

$$\gamma_\ell(x,y) = 4\pi\theta \sum_{m=1}^{\infty} \phi_{m,\ell}(x)\, \phi_{\ell,m}(y)\,. \tag{5.18}$$

The function (5.18) is computed in appendix G. It follows from (G.4) that, at finite N, the propagator (5.17) respects the magnetic translation invariance described in section 2.2. As discussed there, however, this will no longer be true in the desired limit $N \to \infty$ with θ/N finite. The two-point function (5.17) is to be evaluated in the limit $N \to \infty$, $\ell \to \infty$ with $\frac{\ell}{N} \in [0,1]$ fixed. The function $W(\lambda_\ell)$ in this scaling limit is given by (4.33), while the sum over Landau wavefunctions (5.18) is computed in appendix G with the result

$$\gamma_\ell(x,y) = 4\, J_0\Bigl(\Lambda \sqrt{\lambda_\ell}\, |x-y|\Bigr)\,, \tag{5.19}$$

where J_0 is the Bessel function of the first kind of order 0. In the large N limit, the function (5.17) thereby becomes

$$\begin{aligned} G(x,y) &= -\int_0^{16\pi} \frac{\mathrm{d}\lambda}{4\pi}\, W_{\rm b}(\lambda)\, J_0\Bigl(\Lambda \sqrt{\lambda}\, |x-y|\Bigr) \\ &= -\int_0^{4\sqrt{\pi}\,\Lambda} \frac{\mathrm{d}p\ p}{2\pi\Lambda^2}\, W_{\rm b}\left(\frac{p^2}{\Lambda^2}\right)\, J_0\Bigl(p\, |x-y|\Bigr)\,, \end{aligned} \tag{5.20}$$

which by using the angular integral representation of the Bessel function can be written in the form

$$G(x,y) = -\frac{1}{\Lambda^2} \int_{|p| \leq 4\sqrt{\pi}\,\Lambda} \frac{\mathrm{d}^2 p}{(2\pi)^2}\, W_{\rm b}\left(\frac{p^2}{\Lambda^2}\right)\ \mathrm{e}^{\,\mathrm{i}\, p\cdot(x-y)}\,. \tag{5.21}$$

The result (5.21) illustrates, along with the Green's functions of the previous subsection, a remarkable property of the large θ limit of the quantum field theory described in section 3.2. In this limit, the underlying spacetime is expected to disappear or to degenerate and all spacetime symmetries to be maximally violated. Instead, the Green's functions are both rotationally and translationally invariant. This is characteristic of a "fuzzy" regularization scheme, which typically preserves all spacetime symmetries of the original continuum field theory (in contrast to lattice regularization). The remnants of UV/IR mixing here appear in the far infrared at $|x-y| \sim \sqrt{\theta}$. In deriving (5.21), these distances have been effectively scaled out, so that all results here are valid at $|x-y| \ll \sqrt{\theta}$. Thus the spacetime picture which emerges in the limit described in section 3.2 is not pathological.

From (5.21) it also follows that Λ is clearly the ultraviolet cutoff in the quantum field theory. More precisely, the quantity $4\sqrt{\pi}\,\Lambda$ is a sharp cutoff in momentum space. Thus the large θ limit defines a rotationally and translationally invariant field theory with a finite

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cutoff. Moreover, the exact solution of the matrix model Schwinger-Dyson equations has a direct physical meaning as the exact propagator of the quantum field theory. From (5.21) we see that the function (4.33) is essentially the Fourier transform of the exact two-point function,

$$\tilde{G}(p) = -\frac{1}{\Lambda^2}\, W_{\mathrm{b}}\left(\frac{p^2}{\Lambda^2}\right) \tag{5.22}$$

with $|p| \leq 4\sqrt{\pi}\,\Lambda$. The crucial issue now is whether or not the momentum cutoff can be removed, i.e. if the limit $\Lambda \to \infty$ can be taken in the interacting quantum field theory. It is straightforward to see that the two-point function is not finite in the naive scaling limit $\tilde{\mu} = \mu/\Lambda$, $\tilde{g} = g/\Lambda^2$. This can be seen directly in perturbation theory. Using the iterative solution (4.27) and (5.22), we can easily determine the propagator up to two-loop order in the limit $\Lambda \to \infty$ as

$$\begin{aligned}\tilde{G}(p) \;=\; & \frac{1}{p^2+\mu^2} - \frac{g\,\ln(16\pi\,\Lambda^2/\mu^2)}{16\pi\,(p^2+\mu^2)^2} - \frac{g\,\Lambda^2}{(p^2+\mu^2)^3} - \\ & - \frac{g^2}{16\pi\,\mu^2(p^2+\mu^2)^2}\left(\frac{\ln(16\pi\,\Lambda^2/\mu^2)}{16\pi} - \frac{\Lambda^4}{2\mu^4}\right) + \frac{3g^2\,\Lambda^2\ln(16\pi\,\Lambda^2/\mu^2)}{16\pi\,(p^2+\mu^2)^4} - \\ & - \frac{g^2}{16\pi\,(p^2+\mu^2)^3}\left(\frac{\ln^2(16\pi\,\Lambda^2/\mu^2)}{16\pi} + \frac{\Lambda^2}{\mu^2}\right) + \frac{2g^2\,\Lambda^4}{(p^2+\mu^2)^5} + O\left(g^3\right).\end{aligned} \tag{5.23}$$

The first term in (5.23) is the free propagator which as expected is finite as the momentum cutoff is removed. The second term recovers the usual, one-loop logarithmic ultraviolet divergence of Φ^4 theory in two dimensions which is generated by the planar bubble diagram. However, at one-loop order there is an additional quadratic ultraviolet divergence, which may be attributed to the non-planar contribution in this case. The divergences are even worse at higher loop orders. For instance, the usual two-loop ultraviolet divergence generated by the planar sunset diagram is accompanied by terms in (5.23) whose degree of divergence is different than that in the usual scalar field theory. These additional divergences in Λ are nothing but divergences in the summations over degenerate Landau levels in the matrix model, whose degree grows with the order of perturbation theory.

Another way to try to get rid of the cutoff is to adjust the coupling constants non-trivially. For this, one needs to find a critical regime in which there is another scale that is smaller than Λ and the couplings can be thereby adjusted to achieve a separation of scales, such that non-trivial physics remains for momentum modes with $|p| \ll \Lambda$. With the exact solution at hand, it is not difficult to show that this is in fact not possible. For this, we note that from (5.22) it follows that the behaviour of the propagator at large distances (much larger than the inverse momentum cutoff Λ^{-1}) is determined by the singularities of the loop amplitude $W(\lambda)$, i.e. its two complex branch points $\lambda = b_1, b_2$. As a consequence, we can write down the asymptotic behaviour

$$G(x,y) \simeq \mathrm{e}^{-|x-y|/\xi}\,, \qquad |x-y| \gg \Lambda^{-1}\,, \tag{5.24}$$

where the correlation length $\xi = 1/\mathrm{Im}\,p_0$ is determined by the condition that the complex number $z = p_0^2/\Lambda^2$ coincides with one of the two branch points b_1 or b_2. From (4.34) one can show

$$(\operatorname{Im} z)^2 - 2\,(\operatorname{Re} z)^2 = b_1^2\,b_2^2 - \frac{3}{4}\Big(b_1 + b_2\Big)^2 > 2\tilde{g}\,. \tag{5.25}$$

From this inequality one may then show that for any choice of parameters, the correlation length ξ is bounded as

$$\frac{1}{\xi} > \left(\frac{\tilde{g}}{3}\right)^{1/4}\,\Lambda\,. \tag{5.26}$$

From (5.26) it follows that the only way to keep the correlation length finite while sending the cutoff to infinity is to send the coupling constant $\tilde{g}$ to zero. Careful inspection of the function (4.33) shows that the only scaling limit which makes sense is $\tilde{g} \sim 1/\Lambda^4$, which is also suggested by the inequality (5.26). Thus we define

$$g = \frac{m^4}{\Lambda^2} \tag{5.27}$$

with m a dynamically generated mass scale. From the constraint equations (4.34) we further find that $b_1 b_2 = 2\tilde{g}$ and also $b_1 + b_2 = O(\tilde{g})$, which drops out in the scaling limit described above. The renormalized two-point function is thereby given from (4.33) and (5.22) as

$$\tilde{G}(p) = \frac{\sqrt{(p^2+\mu^2)^2 + 4m^4} - (p^2+\mu^2)}{2m^4}\,, \tag{5.28}$$

so that the additional limit $\Lambda \to \infty$ makes the spectral density of the external field invisible in the loop equation (4.25). The meaning of the scaling limit in (5.28) is easy to understand from perturbation theory. Comparing the large momentum expansion of (5.28) with (5.23), it is evident that taking this scaling limit amounts to a non-perturbative resummation of the leading power divergences in perturbation theory. As these divergences come from the Landau level degeneracies and are thereby an artifact of the particular regularization used, such a resummation procedure does not lead to a meaningful interacting quantum field theory. This will be confirmed in the next subsection, wherein we show that all higher-point Green's functions are trivial.

We conclude that the present model is not a renormalizable, interacting quantum field theory, because its coupling constant flows only to the trivial gaussian fixed point. Within our present formulation of the model, there are *a priori* two scales at hand. One is the size $1/\sqrt{B}$ of the orbit in the lowest Landau level, which is the infrared cutoff and which we have taken to be infinitely large. The other is the size $\sqrt{N/B}$ of the largest Landau level that we allow for, which is the ultraviolet cutoff. What we have found above is that there is no intermediate scale in between the two, no matter what one does with the couplings of the model. The fields are thereby correlated on the scale of the cutoff, and the field theory is not renormalizable in the standard sense. Of course, one needs to analyse the possibility that other scalings, besides the standard 't Hooft limit which was described in section 3.2, may lead to non-trivial fixed points of the renormalization group flows. In section 7 we shall address this important issue and describe some ways that a non-trivial interacting quantum field theory may be attained.

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Nevertheless, the present model provides an interesting example of a noncommutative field theory which has a finite cutoff, and which is exactly solvable. In section 7 we will offer an alternative physical interpretation of the regularization of section 3.2 in which Λ is simply the characteristic mass scale of the theory, and for which the correlation functions at finite Λ can be interpreted as those of the full quantum field theoretic limit. The Green's functions display some novel features which are not encountered in ordinary quantum field theory. For example, because the branch points b_1 and b_2 are complex-valued, the propagator exhibits an unusual oscillatory behaviour on top of its exponential decay. This is similar to the Aharonov-Bohm phases, for charged particles in the background magnetic field $B = \theta^{-1}$, that have been observed on top of the usual area law behaviour of Wilson loops in two-dimensional noncommutative Yang-Mills theory [50]. The origin of this behaviour can be understood from the bare Green's functions (5.7), and the free propagators (5.12) and (5.13) for bosons in a finite external magnetic field, where the Aharonov-Bohm phase factors $x \cdot By$ are explicit. Similar behaviours for the Green's functions of scalar field theory on more complicated noncommutative spacetimes have also been observed in [51].

5.3 Connected correlators

To substantiate the claim made in the previous subsection about the triviality of the interacting quantum field theory, in this subsection we show that the four-point Green's function is trivial, i.e. it factorizes into products of propagators in the scaling limit described above, and further argue that this is a generic feature of all higher order correlators (2.12). For this, we expand the four-point function analogously to the propagator to formally get

$$\begin{aligned} G^{(4)}(x,y;z,w) &= \Big\langle \Phi^\dagger(x)\,\Phi(y)\,\Phi^\dagger(z)\,\Phi(w)\Big\rangle \\ &= (4\pi\theta)^2 \sum_{\ell_1,m_1,\dots,\ell_4,m_4=1}^{\infty} \Big\langle M^\dagger_{m_1\ell_1}\,M_{\ell_2 m_2}\,M^\dagger_{m_3\ell_3}\,M_{\ell_4 m_4}\Big\rangle_{E=\tilde{\mathcal{E}}} \times \\ &\quad \times \phi_{\ell_1,m_1}(x)\,\phi_{m_2,\ell_2}(y)\,\phi_{\ell_3,m_3}(z)\,\phi_{m_4,\ell_4}(w)\,. \end{aligned} \tag{5.29}$$

As in the previous subsection, we make sense of (5.29) by restricting the sums over ℓ_j to $1,\dots,N$ and compute the matrix model correlators in the 't Hooft limit. Again, we replace $M \to U \cdot M$, $M^\dagger \to M^\dagger \cdot U^\dagger$ and integrate over all $N \times N$ unitary matrices U. A straightforward computation using the unitary matrix integral formula [49]

$$\begin{aligned} \int_{U(N)} [\mathrm{d}U] U^\dagger_{k_1\ell_1} U_{\ell_2 k_2} U^\dagger_{k_3\ell_3} U_{\ell_4 k_4} = \frac{1}{N^2}\Big[&\delta_{\ell_1\ell_2}\delta_{k_1k_2}\delta_{\ell_3\ell_4}\delta_{k_3k_4} + \delta_{\ell_1\ell_4}\delta_{k_1k_4}\delta_{\ell_2\ell_3}\delta_{k_2k_3} - \frac{1}{N} \times \\ &\times \Big(\delta_{\ell_1\ell_4}\delta_{k_1k_2}\delta_{\ell_2\ell_3}\delta_{k_3k_4} + \delta_{\ell_1\ell_2}\delta_{k_1k_4}\delta_{\ell_3\ell_4}\delta_{k_2k_3}\Big)\Big] \end{aligned} \tag{5.30}$$

yields

$$\begin{aligned} G^{(4)}_N(x,y;z,w) &= \frac{1}{N^2} \sum_{\ell_1,\dots,\ell_4=1}^{N} \Big\langle \Big(M^\dagger M\Big)_{\ell_1\ell_2} \Big(M^\dagger M\Big)_{\ell_3\ell_4}\Big\rangle_{E=\tilde{\mathcal{E}}} \times \\ &\quad \times \Big(\gamma_{\ell_1\ell_2}(x,y)\,\gamma_{\ell_3\ell_4}(z,w) - \frac{1}{N}\,\gamma_{\ell_1\ell_4}(x,w)\,\gamma_{\ell_3\ell_2}(z,y)\Big) + \Big(\longleftrightarrow\Big)\,, \end{aligned} \tag{5.31}$$

where the notation $(\longleftrightarrow)$ indicates to include the same terms but with the spacetime dependence $(x,y);(z,w)$ replaced by $(x,w);(z,y)$, and we have introduced the generalized Landau sums

$$\gamma_{\ell\ell'}(x,y) = 4\pi\theta \sum_{m=1}^{\infty} \phi_{m,\ell}(x)\, \phi_{\ell',m}(y)\,. \tag{5.32}$$

The diagonal elements of the functions (5.32) of course coincide with (5.18), $\gamma_\ell(x,y) = \gamma_{\ell\ell}(x,y)$.

We can now compute the matrix model correlators appearing in (5.31) in terms of the free energy (3.30) as

$$\begin{aligned}\left\langle \left(M^\dagger M\right)_{\ell_1\ell_2} \left(M^\dagger M\right)_{\ell_3\ell_4} \right\rangle_E &= \frac{1}{N^4}\,\frac{1}{Z_N(E)}\,\frac{\partial^2 Z_N(E)}{\partial E_{\ell_2\ell_1}\,\partial E_{\ell_4\ell_3}} \\ &= N^2\,\frac{\partial\,\Xi_N(E)}{\partial E_{\ell_2\ell_1}}\,\frac{\partial\,\Xi_N(E)}{\partial E_{\ell_4\ell_3}} - \frac{\partial^2\,\Xi_N(E)}{\partial E_{\ell_2\ell_1}\,\partial E_{\ell_4\ell_3}}\,.\end{aligned} \tag{5.33}$$

Since the function $\Xi_N(E)$ depends only on the eigenvalues λ_ℓ of the external field E, we can compute its derivatives by replacing E with $E + v/N$ and using Schrödinger perturbation theory to expand it about $v = 0$ (see (4.18)). One finds

$$\begin{aligned}\frac{\partial\,\Xi_N(E)}{\partial E_{\ell m}} &= -\frac{1}{N}\,\delta_{\ell m}\,W(\lambda_\ell)\,, \\ \frac{\partial^2\,\Xi_N(E)}{\partial E_{\ell m}\,\partial E_{\ell' m'}} &= -\delta_{\ell m}\,\delta_{\ell' m'}\,W_2(\lambda_\ell,\lambda_{\ell'}) - \frac{1}{N}\,\delta_{\ell m'}\,\delta_{\ell' m}\,\Big(1-\delta_{\ell m}\Big)\,\frac{W(\lambda_\ell)-W(\lambda_{\ell'})}{\lambda_\ell-\lambda_{\ell'}}\,,\end{aligned} \tag{5.34}$$

where $W(\lambda_\ell)$ is the loop function (4.23) and we have defined

$$W_2(\lambda_\ell,\lambda_{\ell'}) = \frac{1}{N^2}\,\frac{\partial^2 \ln Z[\lambda]}{\partial\lambda_\ell\,\partial\lambda_{\ell'}}\,. \tag{5.35}$$

The matrix correlator (5.33) is thereby given as

$$\begin{aligned}\left\langle \left(M^\dagger M\right)_{\ell_1\ell_2} \left(M^\dagger M\right)_{\ell_3\ell_4} \right\rangle_E &= \delta_{\ell_1\ell_2}\,\delta_{\ell_3\ell_4}\,\Big(W(\lambda_{\ell_1})\,W(\lambda_{\ell_3}) - W_2(\lambda_{\ell_1},\lambda_{\ell_3})\Big) - \\ &\quad - \frac{1}{N}\,\delta_{\ell_1\ell_4}\,\delta_{\ell_2\ell_3}\,\Big(1-\delta_{\ell_1\ell_2}\Big)\,\frac{W(\lambda_{\ell_1})-W(\lambda_{\ell_3})}{\lambda_{\ell_1}-\lambda_{\ell_3}}\,.\end{aligned} \tag{5.36}$$

The crucial point now is the behaviour of the sums over Landau eigenfunctions in (5.32) in the large N limit. In the previous subsection we saw that for $\ell = \ell'$ this sum is of order 1. For $\ell \neq \ell'$, by using the analysis of appendix G it is possible to show that the sum is exponentially small in $|\ell-\ell'|$ for $\ell-\ell' \simeq N$. Putting everything together, the Green's function (5.31) at $N = \infty$ becomes

$$\begin{aligned}G_N^{(4)}(x,y;z,w) &= \frac{1}{N^2}\sum_{\ell,m=1}^{N} W(\lambda_\ell)\,W(\lambda_m)\,\gamma_{\ell\ell}(x,y)\,\gamma_{mm}(z,w) \;+\; \Big(\longleftrightarrow\Big) \;+ O\left(\frac{1}{N}\right) \\ &= G(x,y)\,G(z,w) + G(x,w)\,G(z,y) + O\left(\frac{1}{N}\right)\end{aligned} \tag{5.37}$$

with $G(x,y)$ the two-point Green's function (5.17). The right-hand side of (5.37) is just the "trivial" connected part of the correlator up to order $\frac{1}{N}$, and this proves that the connected four-point Green's function of the field theory is trivial in the large N limit. From the analysis above we believe that this triviality is unavoidable in the 't Hooft limit, as it is simply a consequence of the well-known large N factorization in the matrix model, which implies that all connected matrix correlation functions vanish. In particular, it should be valid for arbitrary interaction potential V. We believe that this is a generic feature that will generalize to all higher-point functions of the model, and so the quantum field theory in the scaling limit obtained in the previous subsection is gaussian , as all connected $2r$-point Green's functions are of order $1/N^{r-1}$.

The computation above shows that the connected part of the four-point Green's function,

$$G^{(4)}_{\rm conn}(x,y;z,w) = G^{(4)}(x,y;z,w) - G(x,y)\,G(z,w) - G(x,w)\,G(z,y)\,, \tag{5.38}$$

is of order $\frac{1}{N}$ in the 't Hooft limit. However, it also shows that one can compute the quantity $N\,G^{(4)}_{\rm conn}(x,y;z,w)$ explicitly in the large N limit, and everything needed to write down a closed formula for it is given above. The non-vanishing leading term of this and higher Green's functions may be of interest when applying this model as a generalized mean field theory of a local quantum field theory, as we discuss in appendix A. It provides the analog of the random phase approximation in conventional mean field theory. A more detailed investigation of these higher-order Green's functions would be interesting but is left for future work.

6. Higher dimensional generalizations

A remarkable feature of the present class of field theories, and the formalism that we have developed to analyse them, is that everything we have said thus far generalizes to *arbitrary* even dimensionality. Namely, the quantum field theory with action (2.10) is exactly solvable for any n. This is quite unlike the situation that one would expect in ordinary quantum field theory. In this section we will describe the higher-dimensional generalization of the exact solution that we have obtained. As we will see, the $2n$-dimensional case can be mapped *exactly* onto the two-dimensional model (This is also observed in [13, 40]). Since much of the machinery is the same as in the two-dimensional case, we will only highlight the essential changes which occur.

Let us first fix some notation. We can rotate to a local coordinate system on $\mathbb{R}^{2n}$ in which the non-degenerate antisymmetric matrix $\theta = (\theta^{ij})$ assumes its canonical skew-diagonal form

$$\theta = \begin{pmatrix} 0 & \theta^1 & & & \\ -\theta^1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta^n \\ & & & -\theta^n & 0 \end{pmatrix} \tag{6.1}$$

with non-zero skew-eigenvalues θ^i, $i = 1, \dots, n$. Similarly, at the special point $B = (B_{ij}) = \theta^{-1}$, which we assume throughout, the antisymmetric matrix B is in its canonical form with skew-eigenvalues $B_i = (\theta^i)^{-1}$. Corresponding to the $i^{\,\rm th}$ skew block, we introduce the complex coordinates

$$z^i = x^{2i-1} + {\rm i}\, x^{2i} \, , \qquad \overline{z^i} = x^{2i-1} - {\rm i}\, x^{2i} \, . \tag{6.2}$$

We will also use a similar notation for momentum space variables, $K_i = (p_{2i-1}, p_{2i})$, $i = 1, \dots, n$.

6.1 Regularization

The hamiltonian $-\mathsf{D}^2$ in the action (2.1) is the sum of n two-dimensional Landau hamiltonians in the magnetic field variables B_i. Its eigenfunctions are therefore given as a product of two-dimensional Landau wavefunctions,

$$\phi_{\boldsymbol{\ell},\boldsymbol{m}}(x) = \prod_{i=1}^n \phi_{\ell_i,m_i}\left(z^i \, , \overline{z^i} \, ; \, B_i\right) , \tag{6.3}$$

where $\phi_{\ell_i,m_i}(z^i, \overline{z^i}\,; B_i)$ are the two-dimensional Landau eigenfunctions of section 3 in the noncommutativity parameter $\theta^i = (B_i)^{-1}$, and the quantum numbers are

$$\boldsymbol{\ell} = (\ell_1, \dots, \ell_n) \, , \qquad \boldsymbol{m} = (m_1, \dots, m_n) \, , \qquad \ell_i, m_i = 1, 2, \dots \, . \tag{6.4}$$

The functions (6.3) form a complete orthonormal basis in $L^2(\mathbb{R}^{2n})$. From the corresponding two-dimensional results of section 3 we obtain immediately the eigenvalue equation

$$-\mathsf{D}^2 \phi_{\boldsymbol{\ell},\boldsymbol{m}} = 4 \sum_{i=1}^n B_i \left(\ell_i - \frac{1}{2}\right) \phi_{\boldsymbol{\ell},\boldsymbol{m}} \, , \tag{6.5}$$

and the star-product projector relation

$$\phi_{\boldsymbol{\ell},\boldsymbol{m}} \star \phi_{\boldsymbol{\ell}',\boldsymbol{m}'} = \frac{1}{\det(4\pi\theta)^{1/4}} \, \delta_{\boldsymbol{m},\boldsymbol{\ell}'} \, \phi_{\boldsymbol{\ell},\boldsymbol{m}'} \tag{6.6}$$

with $\delta_{\boldsymbol{\ell},\boldsymbol{m}} \equiv \prod_i \delta_{\ell_i,m_i}$. An analogous expression holds for the hamiltonian $-\tilde{\mathsf{D}}^2$.

We can now diagonalize the action (2.1) by the expansions

$$\Phi(x) = s \sum_{\boldsymbol{\ell},\boldsymbol{m} \in \mathbb{N}^n} M_{\boldsymbol{\ell},\boldsymbol{m}} \, \overline{\phi_{\boldsymbol{\ell},\boldsymbol{m}}}(x) \, , \qquad \Phi^\dagger(x) = s \sum_{\boldsymbol{\ell},\boldsymbol{m} \in \mathbb{N}^n} \overline{M_{\boldsymbol{\ell},\boldsymbol{m}}} \, \phi_{\boldsymbol{\ell},\boldsymbol{m}}(x) \, , \tag{6.7}$$

where s is a scaling parameter which we will determine below so as to get the appropriate scaling required for exact solvability of the model. We regard $M = (M_{\boldsymbol{\ell},\boldsymbol{m}})$ as a compact operator on the Hilbert space $\mathcal{H} = \mathcal{S}(\mathbb{N}^n)$, with trace ${\rm Tr}_{\,\mathcal{H}}(M) = \sum_{\boldsymbol{m}} M_{\boldsymbol{m},\boldsymbol{m}}$. We will use an obvious matrix notation with $(M^\dagger M)_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \sum_{\boldsymbol{m}} (M^\dagger)_{\boldsymbol{\ell},\boldsymbol{m}} \, M_{\boldsymbol{m},\boldsymbol{\ell}'}$ and $(M^\dagger)_{\boldsymbol{\ell},\boldsymbol{m}} = \overline{M_{\boldsymbol{m},\boldsymbol{\ell}}}$. Then the action (2.1) can be written in the matrix form

$$\tilde{S}_{\rm b} = {\rm Tr}_{\,\mathcal{H}}\Big(\sigma \, M \, \mathcal{E} \, M^\dagger + \tilde{\sigma} \, M^\dagger \, \mathcal{E} \, M + V_0\left(M^\dagger M\right)\Big) \, , \tag{6.8}$$

where the external field is given by

$$\mathcal{E}_{\boldsymbol{\ell},\boldsymbol{m}} = s^2 \left[4 \sum_{i=1}^{n} B_i \left(\ell_i - \frac{1}{2} \right) + \mu^2 \right] \delta_{\boldsymbol{\ell},\boldsymbol{m}} \tag{6.9}$$

and we have defined the renormalized interaction potential

$$V_0(w) = \sum_{k\geq 2} \frac{s^{2k}}{\det(4\pi\theta)^{(k-1)/2}} \frac{g_k}{k} w^k . \tag{6.10}$$

There are many ways to now regularize the noncommutative field theory in the form (6.8) by mapping it onto a finite dimensional matrix model. As we will see later on, these regularizations are all equivalent in the limit where the size of the matrices becomes infinite. The simplest one, which we will call the "naive" regularization, is to restrict each set of Landau quantum numbers to a common finite range $L < \infty$,

$$\ell_i, m_i = 1, \ldots, L , \qquad i = 1, \ldots, n . \tag{6.11}$$

We can then map the integer vectors (6.4) bijectively to single integers ℓ, m as

$$\ell = \sum_{i=1}^{n} L^{i-1} (\ell_i - 1) + 1 , \qquad m = \sum_{i=1}^{n} L^{i-1} (m_i - 1) + 1 . \tag{6.12}$$

The precise form of the mapping (6.12) is not important here, as any other bijection on $\mathbb{N}^n \to \mathbb{N}$ would work equally well, for instance any lexicographic ordering $\boldsymbol{\ell} \mapsto \ell = \mathrm{lex}(\ell_1, \ldots, \ell_n)$ and $\boldsymbol{m} \mapsto m = \mathrm{lex}(m_1, \ldots, m_n)$. With this mapping, we can then identify matrix elements as $M_{\boldsymbol{\ell},\boldsymbol{m}} \equiv M_{\ell m}$, and the regularization above amounts to restricting $\ell, m = 1, \ldots, N$ with

$$N = L^n . \tag{6.13}$$

It should be stressed though that this is *not* the natural regularization. The one leading to a nice $N \to \infty$ limit is the isotropic regularization whereby the one-particle energies are bounded by a momentum scale Λ' as

$$\sum_{i=1}^{n} |B_i| \left(\ell_i - \frac{1}{2} \right) \leq \Lambda'^2 . \tag{6.14}$$

The isotropic limit is essentially obtained by letting the skew-eigenvalues B_i of the magnetic field all approach the same value. However, to get the correct scaling it is more convenient to proceed first via the naive regularization above.

The regularized partition function is then given by the $N \times N$ matrix integral

$$Z_N(\mathcal{E}) = \int \mathrm{D}M \; \mathrm{D}M^\dagger \; \mathrm{e}^{-\operatorname{Tr}\left(\sigma \, M \, \mathcal{E} \, M^\dagger + \tilde{\sigma} \, M^\dagger \, \mathcal{E} \, M + V_0(M^\dagger M) \right)} , \tag{6.15}$$

similarly to that of section 3.2. To obtain a good large N limit, we now need an overall factor of N coming out in front of the trace in (6.15). This requires a relation $s^2 \det(B)^{1/2n} L = 4\pi N = 4\pi L^n$, which fixes the scaling parameter s introduced in (6.7) as

$$s = L^{(n-1)/2} \det(4\pi\theta)^{1/4} . \tag{6.16}$$

The partition function (6.15) then assumes the form in (3.26), with the renormalized interaction potential (6.10) taking the form (3.27). As expected from dimensional analysis, the rescaled coupling constants in (3.27) are now given by

$$\tilde{\mu}^2 = \frac{\mu^2}{\Lambda^2}\,, \qquad \tilde{g}_k = \frac{g_k}{\Lambda^{2n-2(n-1)k}} \tag{6.17}$$

with

$$\Lambda^2 = \frac{L}{\det(4\pi\theta)^{1/2n}}\,. \tag{6.18}$$

The quantity (6.18) is the natural ultraviolet cutoff in any dimension, as then the one-particle energies are bounded as $4\sum_i B_i \ell_i \leq 4bLn = 16\pi n\Lambda^2$ (in the isotropic case where all skew-eigenvalues $B_i = b$ are the same). The one-particle energies are now given by $\tilde{\mathcal{E}}_{\boldsymbol{\ell},\boldsymbol{m}} = \lambda_{\boldsymbol{\ell}}\,\delta_{\boldsymbol{\ell},\boldsymbol{m}}$, with

$$\lambda_{\boldsymbol{\ell}} = 4\pi\left[4\sum_{i=1}^{n}\frac{b_i}{L}\left(\ell_i - \frac{1}{2}\right) + \tilde{\mu}^2\right] \tag{6.19}$$

and

$$b_i = \frac{B_i}{\det(B)^{1/2n}}\,. \tag{6.20}$$

After shifting the eigenvalues (6.19) by the mass as in section 4.3, they are scaled correctly so that the spectral density

$$\rho(\lambda) = \frac{1}{N}\sum_{\boldsymbol{\ell}\in(\mathbb{Z}_L)^n}\delta(\lambda - \lambda_{\boldsymbol{\ell}}) \tag{6.21}$$

is given by a finite Riemann sum in the large N limit,

$$\rho(\lambda) = \prod_{i=1}^{n}\int_0^{16\pi}\frac{\mathrm{d}\lambda_i}{16\pi}\;\delta\left(\lambda - \sum_{i=1}^{n} b_i\lambda_i\right)\,. \tag{6.22}$$

From (6.22) it follows that the isotropic regularization in (6.14), i.e. $\sum_i \ell_i \leq L$, is simpler and more natural. In that case, rather than the limits $\lambda_i \leq 16\pi$ in the integrals defining the density of states (6.22), we have $\sum_i b_i\lambda_i \leq 16\pi$. We can then transform to spherical coordinates to explicitly compute the integral in (6.22) and obtain the large N eigenvalue distribution

$$\rho(\lambda) = \frac{\lambda^{n-1}}{(16\sqrt{\pi}\,)^n\;\Gamma\left(n/2\right)}\,, \tag{6.23}$$

with compact support $\lambda \in [0, 16\pi]$. Here Γ is the Euler function which arises from integrating over the solid angle in n dimensions. Thus one of the main characterizations of the higher dimensional generalizations is that the spectral density is no longer flat.

6.2 The two-point function

Let us now consider the two-point function $G(x,y)$ in the case of the noncommutative field theory (2.10). With the identifications $\boldsymbol{\ell} \equiv \ell$, $\boldsymbol{m} \equiv m$, the matrix model representation (6.7,6.8) is formally the same as in the two-dimensional case, and we thus find

$$G_N(x,y) = -\frac{1}{N} \sum_{\boldsymbol{\ell} \in (\mathbb{Z}_L)^n} W(\lambda_{\boldsymbol{\ell}})\, \gamma_{\boldsymbol{\ell}}(x,y) \tag{6.24}$$

with $W(\lambda_{\boldsymbol{\ell}})$ the logarithmic derivative of the partition function (6.15) in the 't Hooft scaling limit and

$$\gamma_{\boldsymbol{\ell}}(x,y) = \det(4\pi\theta)^{1/2n}\, L^{n-1} \sum_{\boldsymbol{m} \in (\mathbb{Z}_L)^n} \phi_{\boldsymbol{\ell},\boldsymbol{m}}(x)\, \phi_{\boldsymbol{m},\boldsymbol{\ell}}(y)\,. \tag{6.25}$$

Since the Landau eigenfunctions in (6.25) are just the products (6.3) of two-dimensional Landau wavefunctions, the sum in (6.25) is a product of the n sums computed in appendix G and in the scaling limit whereby ℓ/L is finite as $L \to \infty$ it gives the result

$$\begin{aligned} \gamma_{\boldsymbol{\ell}}(x,y) &= \det(4\pi\theta)^{1/2n}\, L^{n-1} \prod_{i=1}^{n} \sum_{m_i=1}^{\infty} \phi_{\ell_i,m_i}\left(z^i\,, \overline{z^i}\,;\, B_i\right)\, \phi_{m_i,\ell_i}\left(w^i\,, \overline{w^i}\,;\, B_i\right) \\ &= 4^n\, \Lambda^{2n-2} \prod_{i=1}^{n} J_0\left(\sqrt{\lambda_i\, b_i}\ \Lambda\ \left|z^i - w^i\right|\right)\ , \end{aligned} \tag{6.26}$$

with $w^i = y^{2i-1} + \mathrm{i}\, y^{2i}$, $\overline{w^i} = y^{2i-1} - \mathrm{i}\, y^{2i}$, and $\lambda_i = 16\pi\ell_i/L$. In the limit $L \to \infty$, the Green's function (6.24) thereby becomes

$$G(x,y) = -\Lambda^{2n-2} \prod_{i=1}^{n} \int_0^{16\pi} \frac{\mathrm{d}\lambda_i}{4\pi}\ J_0\left(\sqrt{\lambda_i\, b_i}\ \Lambda\ \left|z^i - w^i\right|\right)\ W\left(\sum_{i=1}^{n} b_i\lambda_i\right)\,. \tag{6.27}$$

We now change variables $\lambda_i\, b_i\, \Lambda^2 = K_i^2$ in (6.27), use the fact that $\prod_i b_i = 1$, and then apply the angular integral representation of the Bessel function J_0 to get

$$G(x,y) = -\frac{1}{\Lambda^2} \prod_{i=1}^{n} \int_{K_i^2 \leq 16\pi b_i \Lambda^2} \frac{\mathrm{d}K_i\ K_i}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\tau_i}{2\pi}\ \mathrm{e}^{\,\mathrm{i}\, K_i |z^i - w^i| \cos\tau_i}\ W\left(\sum_{i=1}^{n} \frac{K_i^2}{\Lambda^2}\right) \tag{6.28}$$

We can thereby identify $p_{2i-1} = K_i \cos\tau_i$, $p_{2i} = K_i \sin\tau_i$ as the two-dimensional momenta associated with the skew-block coordinates (x^{2i-1}, x^{2i}), and write

$$G(x,y) = -\frac{1}{\Lambda^2} \int_{\left\{p_{2i-1}^2 + p_{2i}^2 \leq 16\pi\Lambda^2 b_i\right\}} \frac{\mathrm{d}^{2n}p}{(2\pi)^{2n}}\ W\left(\frac{p^2}{\Lambda^2}\right)\ \mathrm{e}^{\,\mathrm{i}\, p\cdot(x-y)}\,, \tag{6.29}$$

which is the $2n$-dimensional generalization of the formula (5.21) for the propagator. Thus the scaling limit used above always looks rotationally symmetric except at very small distances. However, this anisotropy is due to the anisotropic regularization that we have used, which is invariant only under an abelian $\mathrm{SO}(2)^n$ subgroup of $\mathrm{SO}(2n)$. Had we used the isotropic scaling described earlier we would have obtained a rotationally symmetric cutoff

$p^2 \leq 16\pi\Lambda^2$ in momentum space. No matter how one chooses the skew-eigenvalues B_i, in the large N limit the anisotropy is totally washed out and one gets the same result. Thus, rather remarkably, we are left with an $\mathrm{SO}(2n)$-invariant Green's function, even though rotational symmetry is broken by the external fields. While other regularizations are possible as well, at finite cutoff Λ they only make a difference at small distances $|x-y| \ll \Lambda^{-1}$.

The Fourier transform $W(\lambda)$ of the two-point function satisfies the same loop equation (4.25) as in the two-dimensional case (before the mass shift), whose solution is given in (4.29), (4.30). The only change is that the density of states $\rho(\lambda)$ is no longer flat, and with the isotropic regularization it is given by the simple explicit expression (6.23). In particular, from the iterative solution (4.27) it follows that the perturbative expansion of the propagator now contains the anticipated power law divergences of scalar field theory in $2n$ dimensions. Moreover, the analytic structure of the loop function $W(\lambda)$ is fairly insensitive to the precise form of the spectral density, as long as its support lies in $[0,\infty)$. In particular, the singularities of $W(\lambda)$ are again cuts which sit in the complex plane, and it is possible to demonstrate once again that the field theory in *any* dimension is not renormalizable. In the limit $\Lambda \to \infty$, the rotational symmetry breaking $\mathrm{SO}(2n) \to \mathrm{SO}(2)^n$ is no longer washed away.

7. Scaling limits

In the previous sections we have found that, in the scaling limit whereby the quantity (3.24) plays the role of an ultraviolet cutoff, the renormalized quantum field theory is trivial in the sense explained before. The purpose of this section is to point out that there exist other scaling limits of the noncommutative field theory which, while mathematically sharing the same properties as that studied already, have drastically different physical interpretations and have the potential of providing non-trivial connected Green's functions. At the same time, we will give nice physical characterizations of the large N limit of the matrix model from a field theoretic point of view. We shall also describe extensions of our analysis that remove the Landau level degeneracies which spoilt the renormalizability of the interacting quantum field theory.

7.1 Scale transformations

The crux of the existence of different scaling limits of the noncommutative quantum field theory resides in the special scaling property of the Landau wavefunctions,

$$\phi_{\ell,m}(x;B) = \lambda^n \ \phi_{\ell,m}\left(\lambda\, x\, ;\, \lambda^{-2}\, B\right) , \tag{7.1}$$

which easily follows from the generating function (3.8). The physical interpretation of (7.1) is that a typical skew-eigenvalue b of B defines the magnetic length $l_{\rm mag} = 1/\sqrt{b}$. If the length scale is changed as $l_{\rm mag} \to \lambda\, l_{\rm mag}$, then the physics is unchanged provided we rescale the spacetime coordinates also as $x \to \lambda\, x$. The factor of λ^n in (7.1) then ensures the correct normalization of the Landau eigenfunctions.

Let us now consider the Green's functions in the regulated field theory at *finite* N,

$$G_N(x_1,\dots,x_r;y_1,\dots,y_r) = \zeta^{2r} \left\langle \Phi^\dagger(x_1)\, \Phi(y_1) \cdots \Phi^\dagger(x_r)\, \Phi(y_r) \right\rangle , \tag{7.2}$$

where the parameter $\zeta = \zeta_N > 0$ is a multiplicative wavefunction renormalization. Then the regularized partition function (6.15) and all Green's functions (7.2) at finite N are invariant under the scale transformations

$$\begin{aligned} x &\longmapsto \lambda\, x\,, \\ B &\longmapsto \lambda^{-2}\, B\,, \\ \theta &\longmapsto \lambda^{2}\, \theta\,, \\ \mu^2 &\longmapsto \lambda^{-2}\, \mu^2\,, \\ g_k &\longmapsto \lambda^{2k(n-1)-2n}\, g_k\,, \\ \zeta &\longmapsto \lambda^{n-1}\, \zeta\,, \end{aligned} \tag{7.3}$$

where we have redefined coordinates using (7.1) so that the scaling parameter $s = 1$ in (6.7)–(6.10). This property is elementary to check by using (7.1) and compensating the transformations (7.3) by the change of matrix integration variables $M \to \lambda\, M$, $M^\dagger \to \lambda\, M^\dagger$. The jacobian λ^{2N^2} of this latter transformation is not important here, as it cancels out in the Green's functions and leads to a finite, irrelevant shift of the free energy.

This scaling property is important for the physical interpretation of the large N limit of the matrix model. It implies that we are free to make the transformation (7.3) with $\lambda = N^\nu$ for arbitrary real ν, before taking the limit $N \to \infty$. The result is independent of the exponent ν. However, even though the mathematical expressions for the Green's functions are unchanged, the physical interpretation of the large N limit does depend on ν. Choosing the scaling $x \to N^\nu\, x$ and sending $N \to \infty$ for $\nu > 0$ means that we blow up small distances and "zoom" into space. On the other hand, for $\nu < 0$ the opposite phenomenon occurs, we "zoom" out so that short distance structures become invisible. The first scaling limit is an infrared limit while the second one is an ultraviolet limit. To be more precise, let us put the system on a hypercubic lattice of side R and spacing a, so that there are $N = (R/a)^{2n}$ lattice points. In conventional quantum field theory, one usually performs the ultraviolet limit, which means sending $a \to 0$ keeping R fixed (and adjusting all parameters so that the limit makes mathematical sense). In the infrared (or thermodynamic) limit, one sends $R \to \infty$ keeping a fixed (so that the finiteness of the spatial volume no longer matters).

This suggests that the parameter Λ introduced in (3.24) has different interpretations dependent on the choice of exponent ν. In the two cases described above we may identify $\Lambda \propto 1/R$ with $a \to 0$ or $\Lambda \propto 1/a$ with $R \to \infty$, respectively. There can also be many other interpolating identifications as well. The definition (3.24) is simply the statement that there is a length scale in the field theory, the renormalized magnetic length. Any field theoretic interpretation of this length is possible, and this point of view is consistent with the duality described in section 2.3. This is one of the various unconventional properties possessed by the present class of quantum field theories. In the ensuing subsections we shall make these statements precise and explicit.

For this, we will introduce the two basic cutoffs above more precisely. The infrared cutoff R is like the diameter of space and is defined by

$$R^2 = 4\pi N' \theta \,, \tag{7.4}$$

with $N' \neq N$. The significance of this length scale is well-known from the physics of the quantum Hall effect. To understand its interpretation better, consider the $\ell = 1$ Landau eigenfunctions which from (3.8) and (3.9) are given explicitly by

$$\phi_{1,m}(x) = \frac{\overline{z}^{\,m-1}}{\sqrt{\pi\theta\,(m-1)!}}\ \mathrm{e}^{\,-|z|^2/2\theta}\ . \tag{7.5}$$

The one-particle "location" $|\phi_{1,m}(x)|^2$ thereby has a sharp peak at $|z| \propto \sqrt{(m-1)\,\theta}$, and so restricting the quantum numbers $m < N'$ is equivalent to the bound $|z| < R$ above. This argument also works for $\ell > 1$. In other words, the infrared cutoff (7.4) amounts to constraining the system quantum numbers to lie below a finite number N' within each Landau level, as in the quantum Hall effect. This cutoff was ignored previously and implicitly sent to infinity.

The other cutoff is our standard ultraviolet cutoff which is the lattice spacing a defined by

$$a^2 = \frac{1}{\Lambda^2} = \frac{4\pi\theta}{N}\ . \tag{7.6}$$

For clarity, in the remainder of this section we shall work mostly in two dimensions $n = 1$ and only with the finite N two-point function which is given by

$$G_N(x,y) = -\sum_{\ell=1}^{N}\ \sum_{m=1}^{\infty} w_N(e_\ell)\ \phi_{m,\ell}(x)\ \phi_{\ell,m}(y)\ . \tag{7.7}$$

Here $e_\ell = 4B(\ell - \frac{1}{2}) + \mu^2$ and $w_N(\xi)$ is a function which converges to the rescaled loop function $\frac{1}{\Lambda^2}\,W(\xi/\Lambda^2)$ of the matrix model in the limit $N \to \infty$. According to (7.3) we may set $\zeta = 1$ in two dimensions. Note that $e_\ell \to p^2 + \mu^2$ at large N.

7.2 Infrared limit

The particular scaling we studied earlier, whereby a finite ultraviolet cutoff is kept, will be refered to as the "infrared limit" of the noncommutative field theory. Explicitly, it is defined by first taking $R \to \infty$, then $N \to \infty$ such that a is finite. Looking at the non-interacting Green's function (5.13), the asymptotic behaviour $K_0(z) \simeq -\ln z$ for $z \to 0$ implies that this limit corresponds to "zooming" into space and the breaking of translation invariance by the magnetic field becomes invisible. In terms of the scale transformations of the previous subsection, it corresponds to setting $\nu = -1$ and rescaling the couplings according to (7.3) with $\lambda = 1/\sqrt{4\pi\theta}$. The quantity (3.24) is thereby interpreted as an ultraviolet cutoff and the Green's function is computed in the limit

$$G_{\rm ir}(x,y) = \lim_{N\to\infty}\ G_N(x,y)\bigg|_{\theta = N/4\pi\Lambda^2} \tag{7.8}$$

keeping Λ finite. The result is given by (5.21), with $\frac{16\pi\ell}{N} \to \frac{p^2}{\Lambda^2}$ in the limit. Everything is known about this limit. The quantum field theory is quasi-free, i.e. everything can be calculated solely from the explicit knowledge of the two-point Green's function. There is no new length scale introduced between $a \to 0$ and ∞.

7.3 Ultraviolet limit

Combining the result above with the duality described in section 2.3 immediately implies the existence of another scaling in the noncommutative field theory in which the physics is completely different. We will refer to this dual scaling limit as the "ultraviolet limit". In fact, it is straightforward to establish the duality property of the finite N Green's function (7.7) for fixed parameters g and θ. This follows directly from the generating functional (3.8), which can be easily checked to possess the symmetries $\mathcal{P}_{s,t}(-x) = \mathcal{P}_{-s,-t}(x)$ and $\tilde{\mathcal{P}}_{s,t}(p) = 2\pi\theta\,\mathcal{P}_{\mathrm{i}\,s,\mathrm{i}\,t}(\theta\,p)$. It follows that the Landau eigenfunctions obey

$$\begin{aligned} \tilde{\phi}_{\ell,m}(p) &= -2\pi\,\mathrm{i}^{\,\ell+m}\,\theta\,\phi_{m,\ell}(\theta\,p)\,, \\ \phi_{\ell,m}(-x) &= (-1)^{\ell+m}\,\phi_{\ell,m}(x)\,, \end{aligned} \tag{7.9}$$

and hence that the regulated propagator has the duality

$$\tilde{G}_N(p,q) = (2\pi\theta)^2\,G_N(\theta\,p\,,\theta\,q) \tag{7.10}$$

under Fourier transformation.

This property implies that the limit

$$\tilde{G}_{\mathrm{uv}}(p,q) = \lim_{N\to\infty}\frac{1}{N^2}\,\tilde{G}_N\left(\frac{p}{N}\,,\frac{q}{N}\right)\bigg|_{\theta=N/4\pi\Lambda^2} \tag{7.11}$$

keeping Λ finite is also well-defined. By using (5.21) and substituting $x = -p/4\pi\Lambda^2$ it can be written as

$$\begin{aligned} \tilde{G}_{\mathrm{uv}}(p,q) &= \frac{1}{4\Lambda^4}\,G\left(\frac{p}{4\pi\Lambda^2}\,,\frac{q}{4\pi\Lambda^2}\right) \\ &= -\frac{1}{\Lambda^2}\int_{|x|\le 1/\sqrt{\pi}\,\Lambda}\mathrm{d}^2x\;W_{\mathrm{b}}\Big((4\pi\Lambda\,x)^2\Big)\;\mathrm{e}^{\,-\,\mathrm{i}\,x\cdot(p-q)}\,, \end{aligned} \tag{7.12}$$

and it thereby gives an ultra-local Green's function in the ultraviolet scaling limit,[6]

$$G_{\mathrm{uv}}(x,y) = -\frac{1}{\Lambda^2}\,W_{\mathrm{b}}\Big((4\pi\Lambda\,x)^2\Big)\;\delta^{(2)}(x-y) \tag{7.13}$$

[6]Note that neither the ultraviolet nor the infrared limits respect magnetic translations. The reason was explained at the beginning of section 2.2. Magnetic translation invariance implies translation invariance in both position and momentum space (up to phase factors). However, the limits $\theta\to\infty$ and $\theta\to 0$ are singular. In the latter case the translational symmetry is broken in position space (and the propagator is ultra-local there), while in the former limit it is broken in momentum space. The breaking of translational invariance also occurs to all orders in perturbation theory of standard noncommutative Φ^4-theory with a finite cutoff [9].

with $|x| \leq 1/\sqrt{\pi}\,\Lambda$. In particular, in the free case it reduces to

$$G^{(0)}_{\rm uv}(x,y) = \frac{1}{(4\pi\Lambda^2\,x)^2+\mu^2}~\delta^{(2)}(x-y)\,. \tag{7.14}$$

It is interesting to note that this scaling limit could also have been defined directly as

$$G_{\rm uv}(x,y) = \lim_{N\to\infty}\,N^2\,G_N(Nx,Ny)\Big|_{\theta=N/4\pi\Lambda^2}~. \tag{7.15}$$

The limit in (7.15) is technically rather difficult to carry out explicitly. But the duality symmetry of the quantum field theory enables a quick and easy derivation from the corresponding results of the infrared limit above.

This scaling corresponds to first taking the limit $a\to0$, then $N'\to\infty$ such that R is finite. In terms of the scale transformations of section 7.1, it corresponds to setting $\nu=1$ and rescaling the couplings according to (7.3) with $\lambda=1/\sqrt{4\pi\theta}$. Note that this means now $\theta\to0$ and Λ is interpreted as a finite infrared cutoff. From (5.13) and the asymptotic behaviour $K_0(z)\simeq\sqrt{\frac{\pi}{2z}}~{\rm e}^{\,-z}$ for $z\to\infty$ it follows that this limit corresponds to "zooming" out of space, i.e. probing large distances of order $N/\Lambda=\sqrt{4\pi N\,\theta}$, and all that is observable is the spatial dependence due to the magnetic field whereas the center of mass dependence becomes ultra-local. Because of the UV/IR duality, this ultraviolet limit is given by the *same* 't Hooft limit of the matrix model, and all that happens is that the long and short distance cutoffs of the theory are interchanged. Everything about this limit is also understood, because its properties are just dual to those of the previous subsection. In particular, the quantum field theory is again quasi-free, but with two-point Green's function given by the ultra-local form (7.13), and there is no new length scale introduced between 0 and $R\to\infty$.

While this argument for going from the infrared to the ultraviolet limit is mathematically very simple, it is somewhat paradoxical. In the infrared limit we take $\theta\to\infty$ and then, by a simple rescaling of the spatial arguments leaving the matrix integral unchanged, we arrive at a limit whereby $\theta\to0$. The resolution of this paradox comes from using the invariance of the matrix integral under $M\to N^\nu\,M$, $M^\dagger\to N^\nu\,M^\dagger$ before the limit $N\to\infty$ is taken. Thus if we consider just the matrix model, then from (7.3) it follows that the only relevant parameters are $\theta\,\mu^2$ and $\theta\,g$. This suggests a different physical interpretation of the infrared limit above which makes the existence of the ultraviolet limit somewhat less paradoxical. The crucial computation for the infrared limit was the sum over Landau levels (5.19) for large ℓ (of order N). However, following the derivation of appendix G through, it is clear from (7.7) that the same result can be obtained also in some limit keeping the noncommutativity parameter θ finite and

$$G_{\rm ir}(x,y) = \lim_{N\to\infty}\,\frac1N\,G_N\left(\frac{x}{\sqrt N}\,,\,\frac{y}{\sqrt N}\right)\Big|_{\theta=1/4\pi\Lambda^2}~. \tag{7.16}$$

The results obtained are exactly as before, only now Λ has a different interpretation. It is simply the inverse of the length scale set by the noncommutativity parameter, and it need not be taken to infinity any longer. In a similar vein, we may write

$$G_{\rm uv}(x,y) = \lim_{N\to\infty} N\,G_N\left(\sqrt N\,x\,,\sqrt N\,y\right)\Big|_{\theta=1/4\pi\Lambda^2}~. \tag{7.17}$$

This makes clear the interpretations above as zooming in versus out of space for the infrared versus ultraviolet limits. It is natural from the point of view of the noncommutative duality.

This discussion motivates the search for alternative scaling limits in between the other two that we have analysed, in which the Green's functions have a non-trivial structure on both short and large distance scales, and whereby the magnetic translation symmetry is preserved. The simplest one uses both infrared and ultraviolet cutoffs (7.4) and (7.6), and corresponds to a scaling exponent $\nu=0$ with $\lambda=1$. It comes from sending $a\to0$ and $R\to\infty$ such that θ is finite, i.e. $N'=N\to\infty$ with θ finite. This can be regarded as a "true" quantum field theory limit, in which both ultraviolet and infrared cutoffs have been removed. It is defined by

$$G_0(x,y) = \lim_{N\to\infty} G_N(x,y)\Big|_{\theta=1/4\pi\Lambda^2}~, \tag{7.18}$$

and it thereby corresponds to keeping all length scales in between the infrared and ultraviolet regimes (no "zooming" in or out). We believe that such intermediate scaling limits will only lead to an interacting quantum field theory if some non-planar limit of the matrix model is used. Such an analysis is beyond the scope of the present work.

7.4 Lifting the degeneracy of Landau levels

The alternative scaling limits of the noncommutative quantum field theory that we have proposed in this section, and in particular the introduction of the intrinsic infrared cutoff (7.4), have the effect of essentially removing the degeneracies from each Landau level. As is known from the theory of the quantum Hall effect, an alternative way to accomplish this is to add a confining electric potential to the Landau hamiltonian, as we did in (2.1). Thus this model has the potential of also providing an exactly solvable interacting quantum field theory. There are several ways that one may proceed in analysing this perturbation of the original field theory. For instance, one may regard $\tilde\sigma$ in (2.1) as a small parameter and simply examine the perturbative corrections in σ to the results above. This is tractable, as we have illustrated how to completely solve the $\tilde\sigma=0$ model. Alternatively, one could pursue a saddle-point analysis of the eigenvalue representation (4.6) in the large N limit. The effective eigenvalue action describes interacting particles in a common self-interaction potential $\tilde V$ but with an electric field dependent on each individual particle. Particle ℓ feels an electric field ℓ/θ. Thus the equilibrium configuration of the particles occurs when they are well-ordered. Permutation symmetry is then broken, and rather than $N!$ identical saddle points, there is just a single one. This is true for very large $\tilde\sigma$, i.e. for equilibrium positions h_ℓ such that $\sigma\,|h_\ell-h_m|\gg\theta$. For small $\tilde\sigma$ the eigenvalues h_ℓ with $\ell\gg1$ accumulate close to the origin and the logarithmic Vandermonde type repulsions at short distances become important. Finally, one can generalize the auxilliary hermitean matrix integral representation of appendix C to this model, and analyse it as described there. The solution of this generalized noncommutative field theory represents an interesting challenge in the search for non-trivial, exactly solvable interacting models.

Another, more difficult way to lift the degeneracy of the Landau levels is to perturb the field theory away from the special point $B = \theta^{-1}$. This of course destroys its $\mathrm{GL}(\infty)$ symmetry and hence its exact solvability. From (3.18) we see that perturbing the action to $B \neq \theta^{-1}$ is like adding hopping terms, giving a model defined on a two-dimensional lattice $\mathbb{N}^2$ with somewhat unusual shifting actions. One way to proceed in solving this lattice model would be to try to map the terms involving the shift matrix (3.20) onto the generators of an auxilliary Heisenberg algebra represented by infinite-dimensional matrices. The total kinetic energy term would then assume a more natural form. It may be that these additional hopping terms produce relevant deformations of the trivial wilsonian renormalization group fixed point which we have found. A similar sort of perturbation expansion has been studied in [9, 26].

Coupling scalar field theories to fermionic field theories with enough supersymmetry is known to be a way to rid the noncommutative quantum field theory of UV/IR mixing [52, 53]. It may be that the fermionic analog of the quantum field theory with action (2.10) contributes power-like divergences which cancel those of the bosonic theory, and therefore that appropriate supersymmetric extensions yield a renormalizable interacting model. Such supersymmetric field theories are the topic of the next section.

8. Supersymmetric extension

In this final section we will formulate a supersymmetric extension of the scalar quantum field theory defined by the action (2.10). We will only present its explicit construction in detail, and leave aside its exact solution for future work. The particularly interesting aspect of this formulation is that non-trivial supersymmetric interactions can only be formulated using the noncommutative star-product. This fact is even apparent, as we will see, at the level of the free supersymmetric action, whereby the supersymmetry transformations are intrinsically operator-valued and hence are parametrized by elements of a noncommutative algebra. We will begin by writing down the analog of (2.1) with $\sigma = 1$, $\tilde{\sigma} = 0$ for relativistic fermions, and then proceed to describe the supersymmetric combination of the two field theories. Throughout this section we work in two spacetime dimensions and at the special point $B = \theta^{-1}$.

8.1 Fermionic models

Consider the noncommutative field theory of a massive two-component Dirac fermion field $\Psi(x)$ in two dimensions under the influence of a constant background magnetic field. The action is

$$S_{\mathrm{f}} = \int \mathrm{d}^2x \, \left[\Psi^\dagger(x) \left(\sigma^i \, \mathsf{D}_i + \mu_{\mathrm{f}}\right) \Psi(x) + V_\star \left(\Psi^\dagger \star \Psi\right)(x)\right] , \tag{8.1}$$

where σ^i, $i=1,2$ are the usual Pauli spin matrices. The Dirac operator can be diagonalized in the complex coordinates (3.1) by expressing it in terms of the ladder operators (3.2) as

$$\sigma^i\,\mathsf{D}_i = \frac{2}{\sqrt{\theta}}\begin{pmatrix} 0 & \mathsf{a} \\ -\mathsf{a}^\dagger & 0 \end{pmatrix} . \tag{8.2}$$

In terms of the Landau eigenfunctions, the orthonormal solutions of the Dirac equation are then given by

$$\psi^{\pm}_{\ell,m}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_{\ell-1,m}(x) \\ \pm\,\mathrm{i}\ \phi_{\ell,m}(x) \end{pmatrix} \tag{8.3}$$

with

$$\sigma^i\,\mathsf{D}_i\,\psi^{\pm}_{\ell,m} = \pm\,2\,\mathrm{i}\,\sqrt{\frac{\ell-1}{\theta}}\ \psi^{\pm}_{\ell,m}\ , \tag{8.4}$$

where we have used (3.17). Here $m=1,2,\dots$, while $\ell=2,3,\dots$.

The case of $\ell=1$ should be treated separately. The $\pm$ eigenfunctions (8.3) are interchanged under the action of the two-dimensional chirality operator, which is the diagonal Pauli matrix σ^3 in the basis of Dirac matrices that we have chosen. In contrast, zero modes of the Dirac equation are all chiral, as only one independent solution for each m exists. They are given by

$$\psi_{1,m}(x) = \begin{pmatrix} 0 \\ \phi_{1,m}(x) \end{pmatrix} \tag{8.5}$$

with

$$\sigma^i\,\mathsf{D}_i\,\psi_{1,m} = 0\,. \tag{8.6}$$

The existence of chiral zero modes of the Dirac operator follows from the index theorem, according to which the number of zero modes of negative chirality minus the number of zero modes of positive chirality is equal to the total magnetic flux divided by 2π. In the present case the total flux is infinite, as is the number of chiral zero modes.

We can now expand the fermion field of (8.1) in the complete basis of Landau wavefunctions to get

$$\Psi(x) = (4\pi\theta)^{1/4}\ \sum_{m=1}^{\infty}\left(\sum_{\ell=2}^{\infty}\ \sum_{s=\pm} F_{s,m\ell}\ \psi^{s}_{\ell,m}(x) + f_m\ \psi_{0,m}(x)\right)\ , \tag{8.7}$$

where $F_\pm = (F_{\pm,m\ell})_{m\geq1\,,\ell\geq2}$ are complex Grassmann-valued matrices, and $\boldsymbol{f} = (f_m)_{m\geq1}$ is a complex Grassmann-valued vector. Note that $F_\pm$ are rectangular matrices, in contrast to the bosonic case. In terms of them, the action (8.1) can be expressed as an infinite, complex fermionic matrix model coupled to an infinite, complex fermionic vector model by using the star-product identities

$$\begin{aligned}
\left(\psi^{s}_{\ell,m}\right)^\dagger \star \psi^{s'}_{\ell',m'} &= \frac{1}{\sqrt{4\pi\theta}}\,\delta^{ss'}\ \delta_{\ell\ell'}\ \phi_{m,m'}\ , \\
\psi^\dagger_{0,m} \star \psi_{0,m'} &= \frac{1}{\sqrt{4\pi\theta}}\,\phi_{m,m'}\ , \\
\left(\psi^{s}_{\ell,m}\right)^\dagger \star \psi_{0,m'} &= \psi^\dagger_{0,m'} \star \psi^{s}_{\ell,m} = 0
\end{aligned} \tag{8.8}$$

which follow easily from (3.12). The action thereby becomes

$$S_{\rm f} = \sum_{s=\pm} {\rm Tr}_{\mathcal{H}} \left(F_s\,\mathcal{K}^s\,F_s^\dagger\right) + \sum_{s=\pm} \sqrt{4\pi\theta}\,\mu_{\rm f}\ {\rm Tr}_{\mathcal{H}} \left(F_s^\dagger\,F_s\right) + \sqrt{4\pi\theta}\,\mu_{\rm f}\,\boldsymbol{f}^\dagger\boldsymbol{f} + \\ + 4\pi\theta\ {\rm Tr}_{\mathcal{H}}\,V\left(\frac{\sum_{s=\pm}F_s\,F_s^\dagger + \boldsymbol{f}\,\boldsymbol{f}^\dagger}{\sqrt{4\pi\theta}}\right), \tag{8.9}$$

where

$$\mathcal{K}^\pm_{\ell\ell'} = \pm\,{\rm i}\ \sqrt{16\pi(\ell-1)}\ \delta_{\ell\ell'}\,. \tag{8.10}$$

8.2 Supersymmetry transformations

We will now combine the fermionic model of the previous subsection with the bosonic model of section 2.1 and define a supersymmetric field theory on a noncommutative phase space. At a first glance, it appears impossible to have supersymmetry in such a field theory, because bosons and fermions interact with an external magnetic field in a different way. Indeed, the spectrum of the free bosonic hamiltonian, $2B(2\ell-1)$, is offset by the quantity $2B$ with respect to the spectrum of the square of the free Dirac operator, $4B(\ell-1)$. However, this offset can be compensated by a shift in the boson mass. To see how this works, let us compute the partition functions for free scalar fields of mass μ and free fermion fields of mass $\mu_{\rm f}$. Up to irrelevant constants, they are given by

$$\left(Z_{\rm b}^{(0)}\right)^{-1} = \prod_{\ell,m=1}^{\infty}\left(\mu^2+4\ell B-2B\right)\,, \\ Z_{\rm f}^{(0)} = \prod_{m=1}^{\infty}\left[\mu_{\rm f}\prod_{\ell>1}\left(\mu_{\rm f}^2+4\ell B-4B\right)\right]. \tag{8.11}$$

If we now take $\mu^2 = \mu_{\rm f}^2 - 2B$, then the functions in (8.11) are almost the same, up to a minor mismatch in their zero mode contributions.

It is straightforward to see that the free action for the complex scalar field and the Dirac fermion field, with masses related as above, is indeed supersymmetric. Using (3.16) at $B\theta = 1$ and (8.2), the action can be rewritten in terms of ladder operators as

$$S_{\rm susy}^{(0)} = \int {\rm d}^2x\ \left[\Phi^\dagger(x)\left(\mathsf{A}^\dagger\mathsf{A}+\mu_{\rm f}^2\right)\Phi(x) + \Psi^\dagger(x)\begin{pmatrix}\mu_{\rm f} & \mathsf{A}\\ -\mathsf{A}^\dagger & \mu_{\rm f}\end{pmatrix}\Psi(x)\right], \tag{8.12}$$

where $\mathsf{A} = 2\sqrt{B}\,\mathsf{a}$. The action (8.12) is invariant under the infinitesimal supersymmetry transformations

$$\delta_\epsilon\Phi = \epsilon^\dagger\begin{pmatrix}\mathsf{A}^\dagger & 0\\ 0 & \mu_{\rm f}\end{pmatrix}\Psi\,, \\ \delta_\epsilon\Phi^\dagger = \Psi^\dagger\begin{pmatrix}\mathsf{A} & 0\\ 0 & \mu_{\rm f}\end{pmatrix}\epsilon\,, \\ \delta_\epsilon\Psi = \begin{pmatrix}\mu_{\rm f} & \mathsf{A}\\ -\mathsf{A}^\dagger & \mu_{\rm f}\end{pmatrix}\begin{pmatrix}\mathsf{A} & 0\\ 0 & \mu_{\rm f}\end{pmatrix}\epsilon\,\Phi\,, \\ \delta_\epsilon\Psi^\dagger = \Phi^\dagger\,\epsilon^\dagger\begin{pmatrix}\mathsf{A}^\dagger & 0\\ 0 & \mu_{\rm f}\end{pmatrix}\begin{pmatrix}\mu_{\rm f} & \mathsf{A}\\ -\mathsf{A}^\dagger & \mu_{\rm f}\end{pmatrix}, \tag{8.13}$$

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where the parameters ϵ and $\epsilon^\dagger$ of the supersymmetry transformations are arbitrary Grassmann odd functions of the b and $\mathsf{b}^\dagger$ operators in (3.2). Here we use the convention that the action of a differential operator from the right is consistent with integration by parts, i.e.

$$f\,\mathsf{a} \equiv \mathsf{b}^\dagger\, f\,, \qquad f\,\mathsf{a}^\dagger \equiv \mathsf{b}\, f\,. \tag{8.14}$$

It follows that the supersymmetry transformations in this case contain infinitely many parameters. This resembles local supersymmetry somewhat, except that here the parameters of the transformation are arbitrary functions of differential operators rather than arbitrary functions of the spacetime coordinates. A closer analogy is the type of supersymmetry that arises in zero-dimensional supersymmetric matrix models [54]–[56], in which the parameter of the transformation is an arbitrary matrix. In fact, the parameters of the supersymmetry transformation in the present case become matrices after expanding in the basis of Landau eigenfunctions. For this, we note that the kinetic energy operator for the boson field in (8.12) can be written in matrix form as $\mathcal{K}^+\mathcal{K}^-$, where $\mathcal{K}^\pm$ are the matrices defined in (8.10). The action (8.12) can thereby be written as the supersymmetric matrix-vector model

$$\begin{aligned} S_{\rm susy}^{(0)} = {\rm Tr}_{\,\mathcal{H}}\left(\sum_{s=\pm} F_s\left(\mathcal{K}^s + \sqrt{4\pi\theta}\,\mu_{\rm f}\right)F_s^\dagger + M^\dagger\left(\mathcal{K}^+\mathcal{K}^- + 4\pi\theta\,\mu_{\rm f}^2\right)M\right) + \\ + \sqrt{4\pi\theta}\,\mu_{\rm f}\,\boldsymbol{f}^\dagger\boldsymbol{f} + 4\pi\theta\,\left(\mu_{\rm f}^2 - 2\theta^{-1}\right)\boldsymbol{\beta}^\dagger\boldsymbol{\beta}\,, \end{aligned} \tag{8.15}$$

where for convenience we have separated out the zero-mode part $\beta_m \equiv M_{1m}$ of the scalar field, and in (8.15) it is understood that $M \equiv (M_{\ell m})_{\ell\geq 2\,,m\geq 1}$ is a rectangular matrix. The action (8.15) is invariant under the infinitesimal supersymmetry transformations

$$\begin{aligned} \delta_\epsilon F_s &= -\epsilon_s\, M\,\left(\mathcal{K}^{-s} + \sqrt{4\pi\theta}\,\mu_{\rm f}\right)\,, \\ \delta_\epsilon F_s^\dagger &= -\left(\mathcal{K}^{-s} + \sqrt{4\pi\theta}\,\mu_{\rm f}\right)\,M^\dagger\,\epsilon_s^\dagger\,, \\ \delta_\epsilon M &= \sum_{s=\pm}\epsilon_s\,F_s^\dagger\,, \\ \delta_\epsilon M^\dagger &= \sum_{s=\pm} F_s\,\epsilon_s^\dagger\,, \end{aligned} \tag{8.16}$$

where $\epsilon_\pm \equiv (\epsilon_{\pm,\ell\ell'})_{\ell,\ell'\geq 2}$ is a pair of infinite dimensional matrices. As expected, the supersymmetry transformations do not involve zero modes.

Since $\mathcal{K}^+ = (\mathcal{K}^-)^\dagger$, we have $(\delta_\epsilon F_s)^\dagger \neq \delta_\epsilon F_s^\dagger$, and hence F_s and $F_s^\dagger$ should be regarded as independent variables in the functional integral defining the corresponding quantum field theory. The supersymmetry transformation (8.16) is then a legitimate change of variables, and it leads to the supersymmetric Ward identities in the usual way, as long as the integration measure is superinvariant. The condition for this is

$${\rm Tr}_{\,\mathcal{H}}\left(\epsilon_\pm + \epsilon_\pm^\dagger\right) = 0\,. \tag{8.17}$$

The absence of zero modes in (8.16) and the constraint (8.17) eliminate only a finite number of degrees of freedom.

8.3 Supersymmetric interactions

Thus far we have considered a non-interacting supersymmetric quantum field theory for which noncommutativity played no role. Noncommutativity becomes important if we try to answer the question of whether or not there exist interactions that preserve the supersymmetry. We believe that the affirmative answer to this question can be given *only* within the framework of noncommutative field theory. It seems impossible to construct local interactions that are invariant under the supersymmetry transformations (8.13), though we cannot prove this fact rigorously. However, "star-local" interacting quantum field theories which possess the infinite-parameter supersymmetry (8.13) do exist, as we now proceed to demonstrate.

The reason why noncommutativity aids the construction of lagrangians which are invariant under the supersymmetry transformations of the previous subsection is that the ladder operators (3.2) obey remarkable Leibniz-type rules with respect to the star-product,

$$\begin{aligned} \mathsf{a}\,(f\star f') &= (\mathsf{a}\,f)\star f'\,, \\ \mathsf{a}^\dagger\,(f\star f') &= (\mathsf{a}^\dagger\,f)\star f'\,, \\ \mathsf{b}\,(f\star f') &= f\star(\mathsf{b}\,f')\,, \\ \mathsf{b}^\dagger\,(f\star f') &= f\star(\mathsf{b}^\dagger\,f')\,, \\ (\mathsf{b}\,f)\star f' &= f\star(\mathsf{a}^\dagger\,f')\,, \\ (\mathsf{b}^\dagger\,f)\star f' &= f\star(\mathsf{a}\,f')\,. \end{aligned} \tag{8.18}$$

These equalities can be proven by expanding the fields f and f' in the Landau basis and using (3.12). Alternatively, the star-product projector relation (3.12) for the Landau wavefunctions can be derived from the Leibniz rules (8.18), which can be independently checked by a direct computation in the Fourier basis by using (2.5). Formally, the relations (8.18) simply reflect the fact the algebra of functions on $\mathbb{R}^2$, equipped with the star-product, generates a bimodule for the harmonic oscillator algebras (3.3) realized by the commuting a and b operators. This was already implicit in the definitions (8.14).

As usual, the construction of supersymmetric lagrangians is facilitated by the introduction of auxilliary fields $\mathcal{F}$. With them, we postulate the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon\Phi &= \epsilon^\dagger\begin{pmatrix}\mathsf{A}^\dagger & 0\\ 0 & \zeta\end{pmatrix}\Psi\,, \\ \delta_\epsilon\Phi^\dagger &= \Psi^\dagger\begin{pmatrix}\mathsf{A} & 0\\ 0 & \zeta\end{pmatrix}\epsilon\,, \\ \delta_\epsilon\Psi &= \begin{pmatrix}0 & \mathsf{A}\\ -\mathsf{A}^\dagger & 0\end{pmatrix}\begin{pmatrix}\mathsf{A} & 0\\ 0 & \zeta\end{pmatrix}\epsilon\,\Phi - \mathrm{i}\begin{pmatrix}\mathsf{A} & 0\\ 0 & \zeta\end{pmatrix}\epsilon\,\mathcal{F}\,, \\ \delta_\epsilon\Psi^\dagger &= \Phi^\dagger\,\epsilon^\dagger\begin{pmatrix}\mathsf{A}^\dagger & 0\\ 0 & \zeta\end{pmatrix}\begin{pmatrix}0 & \mathsf{A}\\ -\mathsf{A}^\dagger & 0\end{pmatrix} - \mathrm{i}\,\mathcal{F}^\dagger\,\epsilon^\dagger\begin{pmatrix}\mathsf{A}^\dagger & 0\\ 0 & \zeta\end{pmatrix}\,, \\ \delta_\epsilon\mathcal{F} &= \mathrm{i}\,\epsilon^\dagger\begin{pmatrix}\mathsf{A}^\dagger & 0\\ 0 & \zeta\end{pmatrix}\begin{pmatrix}0 & \mathsf{A}\\ -\mathsf{A}^\dagger & 0\end{pmatrix}\Psi\,, \end{aligned}$$

$$\delta_\epsilon \mathcal{F}^\dagger = \mathrm{i}\, \Psi^\dagger \begin{pmatrix} 0 & \mathsf{A} \\ -\mathsf{A}^\dagger & 0 \end{pmatrix} \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \zeta \end{pmatrix} \epsilon\,, \tag{8.19}$$

where ζ is an arbitrary parameter of mass dimension 1. It can be set to unity, but then the left and right components ϵ and $\epsilon^\dagger$ will have different scaling dimensions.

There are two star-quadratic invariants of the supersymmetry transformations (8.19) which contain at most two derivatives of the fields. They are given by

$$\begin{aligned} \mathcal{L}_0 &= \Psi^\dagger \star \begin{pmatrix} 0 & \mathsf{A} \\ -\mathsf{A}^\dagger & 0 \end{pmatrix} \Psi + \Phi^\dagger \star \mathsf{A}^\dagger \mathsf{A}\, \Phi + \mathcal{F}^\dagger \star \mathcal{F}\,, \\ \mathcal{L}_1 &= \Psi^\dagger \star \Psi + \mathrm{i}\, \mathcal{F}^\dagger \star \Phi + \mathrm{i}\, \Phi^\dagger \star \mathcal{F}\,. \end{aligned} \tag{8.20}$$

The superinvariance of the expressions (8.20) depends crucially on the bimodule properties (8.18) of the ladder operators. We can now proceed to construct supersymmetric lagrangians by taking star-products of $\mathcal{L}_1$ and $\mathcal{L}_0$. The most general renormalizable lagrangian, i.e. the one with no couplings of negative dimension, is then given by

$$\mathcal{L} = \mathcal{L}_0 + \mu_{\mathrm{f}}\, \mathcal{L}_1 + \frac{g}{2}\, \mathcal{L}_1 \star \mathcal{L}_1\,. \tag{8.21}$$

The auxilliary fields can be eliminated from (8.21) by using the equations of motion. Then the second term gives masses to the boson and fermion fields, while the third term induces various interactions. These interactions are star-local, but non-polynomial in Φ and Ψ. Explicitly, one finds

$$\begin{aligned} S_{\mathrm{susy}} = \int \mathrm{d}^2x\, \Big[& \Phi^\dagger(x) \left(\mathsf{A}^\dagger \mathsf{A} + \mu_{\mathrm{f}}^2 \right) \Phi(x) + \Psi^\dagger(x) \begin{pmatrix} \mu_{\mathrm{f}} & \mathsf{A} \\ -\mathsf{A}^\dagger & \mu_{\mathrm{f}} \end{pmatrix} \Psi(x) + \\ & + 2g\mu_{\mathrm{f}}^2\, \Phi^\dagger \star \Phi \star \Phi^\dagger \star \Phi(x) + 2g\mu_{\mathrm{f}}\, \Phi^\dagger \star \Phi \star \Psi^\dagger \star \Psi(x) + \\ & + \frac{g}{2}\, \Psi^\dagger \star \Psi \star \Psi^\dagger \star \Psi(x) + O\left(g^2\right) \Big]\,. \end{aligned} \tag{8.22}$$

The action (8.22) possesses the usual star-local $\mathrm{GL}(\infty)$ symmetry as before and hence yields a potentially solvable supersymmetric quantum field theory. It would be interesting to seek a superspace formulation of this model, which may aid in finding its exact solution. However, it is not clear what the concept of a superspace could mean in the present context. The supersymmetry here acts as a fermionic rotation on the Landau levels, which has nothing in common with ordinary supersymmetry. In particular, the commutator of supercharges is not a spacetime translation [55, 56]. The definition of noncommutative superspaces has been addressed recently within various different contexts in [57]–[66]. In the present case, supersymmetric rotations are the super-analogs of the bosonic $\mathrm{GL}(\infty)$ symmetry, so it is tempting to speculate that together they generate an infinite dimensional $\mathrm{GL}(\infty|\infty)$ supergroup. Geometrically, this noncommutative supersymmetry would then correspond to the superspace generalization of area-preserving diffeomorphisms.

Acknowledgments

This paper is dedicated to the memory of Ian Kogan, who will be deeply missed as both a good friend and physicist. We thank M. Abou-Zeid, G. Akemann, A. Alekseev,

J. Ambjørn, H. Braden, N. Dorey, V. Elias, A. Gorsky, H. Grosse, P. Horváthy, D. Johnston, D. O'Connor, V. Kazakov, I. Kogan, D. Mateos, A. Orlov, J.-H. Park and T. Turgut for helpful discussions and correspondence. E.L. and K.Z. would like to thank the Erwin Schrödinger Institute in Vienna, and E.L. would like to thank the Feza Gursey Institute in Istanbul, for hospitality during part of this work. This work was supported in part by the Swedish Science Research Council (VR) and the Göran Gustafssons Foundation. The work of R.J.S. was supported in part by an Advanced Fellowship from the Particle Physics and Astronomy Research Council (U.K.). The work of K.Z. was supported in part by RFBR grant 01–01–00549 and grant 00–15–96557 for the promotion of scientific schools.

A. Mean field analysis of commutative complex Φ^4-theory

In this appendix we point out that the models studied in this paper also have a concrete physical motivation as a novel kind of mean field theory for conventional (commutative) complex Φ^4-theory. For this, we consider the action $S_0 + S_{\rm int}$ for ordinary two-dimensional complex Φ^4-theory written in momentum space, with free part

$$S_0 = \int \mathrm{d}^2k \ \left(k^2 + \mu^2\right) \ \tilde{\Phi}^\dagger(k)\,\tilde{\Phi}(k)\,, \tag{A.1}$$

and the interaction

$$S_{\rm int} = \frac{g_0}{2} \int \mathrm{d}^2k \ \mathrm{d}^2k' \ \mathrm{d}^2q \ \mathrm{d}^2q' \ \delta^{(2)}(k'-k+q'-q)\,\tilde{\Phi}^\dagger(k')\,\tilde{\Phi}(k)\,\tilde{\Phi}^\dagger(q')\,\tilde{\Phi}(q) \tag{A.2}$$

describing all possible momentum conserving processes whereby particles with incoming momenta k and q are scattered to new momenta k' and q'. Standard mean field theory can be applied to this theory by truncating the interaction to retain only the Hartree and Fock terms. In the former interactions the individual particle momenta are conserved, $k = k'$ and $q = q'$, while in the latter terms they are exchanged, $k = q'$ and $q = k'$. In the present case these two types of terms are actually the same, and their sum yields the Hartree-Fock interaction

$$S_{\rm HF} = g_0 \int \mathrm{d}^2k \ \mathrm{d}^2q \ \tilde{\Phi}^\dagger(k)\,\tilde{\Phi}(k)\,\tilde{\Phi}^\dagger(q)\,\tilde{\Phi}(q)\,. \tag{A.3}$$

The quantum field theory with action $S_0 + S_{\rm HF}$ can be solved exactly, and its exact solution is identical to mean field theory for commutative complex Φ^4-theory. Up to now our discussion could have been easily extended to arbitrary spacetime dimensionality, but in two dimensions there are special mixed interactions which are of Hartree type in one component of the momenta and of Fock type in the other component, $k' = (k_1', k_2') = (k_1, q_2)$ and $q' = (q_1', q_2') = (q_1, k_2)$, or vice versa [67]. The sum of all of these mixed terms yields the interaction

$$S_{\rm mixed} = g_0 \int \mathrm{d}^2k \ \mathrm{d}^2q \ \tilde{\Phi}^\dagger(k_1, q_2)\,\tilde{\Phi}(k_1, k_2)\,\tilde{\Phi}^\dagger(q_1, k_2)\,\tilde{\Phi}(q_1, q_2)\,. \tag{A.4}$$

We now claim that the model with action $S_0 + S_{\rm mixed}$ is equivalent to the one studied in this paper. For this, we introduce a regularization by restricting the momenta to $k =$

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$(k_1,k_2)=\frac{2\pi}{R}\,(\ell,m)$ with $\ell,m=0,\pm\,1,\pm\,2,\dots,\pm\,R/a\equiv(N-1)/2$, where $R<\infty$ is a large but finite radius for the size of spacetime serving as an infrared cutoff, and $a>0$ is a small but finite lattice spacing serving as an ultraviolet cutoff. We can then identify the Fourier modes of the fields with the elements of an $N\times N$ matrix $M=(M_{\ell m})$ as

$$M_{\ell m}=\left(\frac{2\pi}{R}\right)^2\,\tilde{\Phi}\left(\frac{2\pi\ell}{R},\frac{2\pi m}{R}\right). \tag{A.5}$$

By introducing a diagonal matrix E with elements

$$E_{\ell m}=\ell^2\;\delta_{\ell m}\,, \tag{A.6}$$

we may thereby write the regulated action as a complex matrix model

$$S_0+S_{\rm mixed}={\rm Tr}\,\left(\sigma\,M^\dagger\,E\,M+\tilde{\sigma}\,M\,E\,M^\dagger+\mu^2\,M^\dagger M+\frac{g}{2}\,M^\dagger MM^\dagger M\right) \tag{A.7}$$

with $\sigma=\tilde{\sigma}=1$, $g=2g_0/R^2$, and Tr the usual $N\times N$ matrix trace as before.

The matrix integral that we evaluate in Section 4.1 provides an exact and explicit formula for the free energy of this model. In Sections 4 and 5 we find the complete solution of the model in the limit $R\to\infty$ for the special maximally anisotropic case $\tilde{\sigma}=0$. In section 7 we examine other possible ways of removing the cutoffs in the anisotropic model. The physical interpretation of the model presented here gives a strong motivation for extending the results of this paper to the more complicated isotropic case where $\sigma=\tilde{\sigma}$.

B. UV/IR duality versus T-duality

Combining the translation and symplectomorphism invariances of the action (2.10) with its Morita-type duality symmetry found in [12] lends further support to the observation of [12] that the quantum field theory defined by (2.10) may be regarded as a discrete noncommutative $\mathbb{Z}_2$ gauge theory when $B=\theta^{-1}$. For this, we introduce a constant, background metric $H=(H_{\mu\nu})$ on $\mathbb{R}^{2n}$ and use it to rewrite the action (2.10) in a covariant form. We regard the action $S_{\rm b}$ as a functional of the dynamical field Φ, the background fields H and B, and the coupling parameters g and θ. The action then has a duality under Fourier transformation [12],

$$S_{\rm b}[\Phi;H,B,g,\theta]=\left|\det\left(\frac{B}{2\pi}\right)\right|\;S_{\rm b}\left[\tilde{\Phi};\tilde{H},\tilde{B},\tilde{g},\tilde{\theta}\right], \tag{B.1}$$

where the dual parameters in momentum space are given by

$$\begin{aligned}\tilde{H}&=B^{-1}\,H\,B^{-1}\,,\\ \tilde{B}&=B^{-1}\,,\\ \tilde{g}&=\frac{g}{\Big|\det(2\pi\theta)\Big|}\,,\\ \tilde{\theta}&=\theta^{-1}\,.\end{aligned} \tag{B.2}$$

Note that, in contrast to the transformations of Section 2.3, here we do not rescale the arguments of the fields.

There is a novel gauge Morita equivalence interpretation of this noncommutative duality, wherein we may heuristically regard the model (2.10) as a noncommutative gauge theory defined along $2n$ finite discrete directions, i.e. on a two-sheeted manifold $\mathbb{R}^{2n} \times \mathbb{Z}_2$. Noncommutative Yang-Mills theory defined on a $2n$-dimensional *torus* is manifestly invariant under the standard $\mathrm{SO}(2n, 2n, \mathbb{Z})$ open string T-duality transformations [3]

$$\begin{aligned} \tilde{\theta} &= (\mathsf{A}\,\theta + \mathsf{B})^\top\,(\theta\,\mathsf{Q} - \mathsf{N})^{-1}\,, \\ \tilde{H} &= (\theta\,\mathsf{Q} - \mathsf{N})^\top\,H\,(\theta\,\mathsf{Q} - \mathsf{N})\,, \\ \tilde{g}_{\mathrm{YM}}^2 &= g_{\mathrm{YM}}^2\ \det(\theta\,\mathsf{Q} - \mathsf{N})\,, \end{aligned} \tag{B.3}$$

where the superscript $\top$ denotes transposition, g_{YM} is the Yang-Mills coupling constant, and in this equation θ is the dimensionless noncommutativity parameter. The $2n \times 2n$ symmetric integral matrix N is proportional to the rank N of the gauge theory, while the antisymmetric integral matrix Q is determined by the magnetic fluxes of the gauge bundle around the various two-cycles of the torus. The symmetric and antisymmetric integral matrices A and B, respectively, are chosen to solve the generalized Diophantine equation

$$\mathsf{A}\,\mathsf{N} + \mathsf{B}\,\mathsf{Q} = \mathbb{1}_{2n}\,. \tag{B.4}$$

Going back to the scalar field theory we may, in the usual Connes-Lott type interpretation of quartic scalar field theory [68], regard Φ as the off-diagonal components of a superconnection, and the field theory (2.10) as induced by the limit in which the diagonal gauge degrees of freedom are frozen out in the usual noncommutative Yang-Mills action for the superconnection. The noncommutative scalar field theory is then formally the zero rank limit of a noncommutative gauge theory. All formulas above still make mathematical sense in this limit and it is heuristically the choice that should be made for a discrete gauge theory. It means that one should set

$$\mathsf{N} = 0\,. \tag{B.5}$$

The equation (B.4) is then solved by

$$\mathsf{A} = 0\,, \qquad \mathsf{Q} = -\mathsf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1}_n\,. \tag{B.6}$$

Then the transformation rules (B.3) coincide exactly with (B.2) at the special point $B = \theta^{-1}$. This holds with the identification $g_{\mathrm{YM}}^2 = ((2\pi)^n\,g)^{-1}$. For this “discrete” truncation of the toroidal compactification, the choice of flux matrix Q in (B.6) is very natural, given the charges carried by the scalar fields Φ (see (2.33)). This gauge theoretic interpretation also indicates why the duality property is uniquely possessed by a quartic interaction of the scalar field, which has other special features as well which are displayed in the main text. It would be interesting to pursue this gauge theoretic interpretation further using the usual techniques of noncommutative geometry, as it may help elucidate further features of the model and its generalizations.

C. Alternative derivation of the loop equations

In this appendix we will present another derivation of the Schwinger-Dyson equations (4.25) for the matrix model (3.26) with potential (2.9). This derivation has the advantage of being somewhat more amenable to generalization to other noncommutative complex Φ^4 field theories, such as that associated with the matrix model (4.1). It also gives another physical interpretation to the loop function (4.23) in terms of the spectral distribution of an auxilliary hermitean one-matrix model.

We introduce an auxilliary $N\times N$ hermitean matrix $X=(X_{\ell m})$ through the Hubbard-Stratonovich transformation

$$\mathrm{e}^{-\frac{g}{2}\,\mathrm{Tr}\,(MM^\dagger)^2}=\int \mathrm{D}X~~\mathrm{e}^{-\mathrm{Tr}\,(\frac{1}{2g}\,X^2+\mathrm{i}\,XMM^\dagger)}\,, \tag{C.1}$$

where $\mathrm{D}X=\prod_\ell \mathrm{d}X_{\ell\ell}~\prod_{m<\ell}\mathrm{d}\,\mathrm{Re}\,X_{\ell m}~\mathrm{d}\,\mathrm{Im}\,X_{\ell m}/2\pi$. After a rescaling of X and M by the matrix dimension N, the partition function (3.26) may then be written in the form

$$Z_N(E)=\int \mathrm{D}M~\mathrm{D}M^\dagger~\int \mathrm{D}X~~\mathrm{e}^{-\frac{N}{2\tilde g}~\mathrm{Tr}~X^2-\vec M^\dagger(\mathbb{1}\otimes E^\top+\mathrm{i}\,X\otimes\mathbb{1})\vec M}\,, \tag{C.2}$$

where for convenience we have changed variables from $N\times N$ complex matrices $M=(M_{\ell m})$ to vectors $\vec M\in\mathbb{C}^{N^2}$ through the rule

$$\vec M=(M_{11},M_{12},\dots,M_{1N},M_{21},\dots,M_{NN})^\top\,. \tag{C.3}$$

The gaussian integration over M and $M^\dagger$ can now be carried out in this manner, giving the effective hermitean one-matrix integral with a Penner type interaction potential,

$$Z_N(E)=\int \mathrm{D}X~~\mathrm{e}^{-\frac{N}{2\tilde g}~\mathrm{Tr}~X^2-\mathrm{Tr}_{~\mathrm{Ad}}~\ln(\mathbb{1}\otimes E^\top+\mathrm{i}\,X\otimes\mathbb{1})}\,, \tag{C.4}$$

where $\mathrm{Tr}_{~\mathrm{Ad}}$ is the matrix trace in the adjoint representation.

The matrix integral (C.4) is invariant under unitary transformations of both X and E, and can thus be written as an eigenvalue model. For this, we diagonalize $X=U\,\mathrm{diag}(\mu_1,\dots,\mu_N)\,U^\dagger$, with $\mu_\ell\in\mathbb{R}$ and $U\in U(N)$, compute the jacobian of the integration measure to get $\mathrm{D}X=[\mathrm{d}U]~\prod_\ell \mathrm{d}\mu_\ell~\Delta_N[\mu]^2$, and thereby write (C.4) as

$$Z_N(E)=\prod_{\ell=1}^N~\int_{-\infty}^\infty \mathrm{d}\mu_\ell~~\mathrm{e}^{-\frac{N}{2\tilde g}\,\mu_\ell^2}~\prod_{m=1}^N\frac{1}{\lambda_m+\mathrm{i}\,\mu_\ell}~\Delta_N[\mu]^2\,. \tag{C.5}$$

The saddle point equation for the large N limit of the eigenvalue integral (C.5) is

$$\frac{\mu_\ell}{\tilde g}-\frac1N\sum_{m\neq\ell}\frac{1}{\mu_\ell-\mu_m}+\frac1N\sum_{m=1}^N\frac{1}{\mu_\ell-\mathrm{i}\,\lambda_m}=0\,. \tag{C.6}$$

We now multiply the expression (C.6) through by $\frac1N\,(\mu_\ell-z)^{-1}$ for $z\in\mathbb{C}$, sum over all $\ell=1,\dots,N$, and expand the terms to rewrite them in terms of the resolvent function

$$\Sigma(z)=\frac1N\sum_{\ell=1}^N\frac{1}{\mu_\ell-z}\,. \tag{C.7}$$

By rewriting the sums over the eigenvalues λ_m of the external field E as integrals over the spectral density (4.24), and dropping the term $\Sigma'(z)/N$ at $N=\infty$, we may in this way write (C.6) as the large N loop equation

$$\frac{1}{\tilde{g}}\Big(1+z\,\Sigma(z)\Big)+\Sigma^2(z)+\int_{a_1}^{a_2}\mathrm{d}\lambda~\frac{\rho(\lambda)}{z+\mathrm{i}\,\lambda}\Big(\Sigma(z)-\Sigma(-\,\mathrm{i}\,\lambda)\Big)=0\,. \tag{C.8}$$

The loop equation (C.8) is essentially the same as (4.25). Since the resolvent function (C.7) has the asymptotic behaviour $\Sigma(z)\simeq-\frac{1}{z}+O(\frac{1}{z^2})$ for $z\to\infty$, it then follows that it may be identified with the function (4.23) through

$$\Sigma(z)=-\,\mathrm{i}\,W(-\,\mathrm{i}\,z)\,. \tag{C.9}$$

This coincidence is not surprising, as both functions W and Σ are the generating functions of correlators of derivatives of $M^\dagger M$ acting in the complex matrix model. Indeed, a straightforward calculation using the Hubbard-Stratonovich transformation above shows

$$\delta_{\ell\ell'}~W(\lambda_\ell)\equiv\frac{1}{N}\Big\langle\Big(M^\dagger M\Big)_{\ell\ell'}\Big\rangle_E=\delta_{\ell\ell'}~\left\langle\frac{1}{N}\sum_{m=1}^{N}\frac{1}{\lambda_\ell+\mathrm{i}\,\mu_m}\right\rangle_{\mathrm{HS}}\,, \tag{C.10}$$

where the second expectation value is taken with respect to the Hubbard-Stratonovich representation (C.4). The usefulness of this approach, however, is that the eigenvalue model (C.5) may be straightforwardly generalized to include also the external field $\tilde{E}$ that appears in the matrix model (4.1), and its loop equations may be analysed by the technique presented here. The exact solvability of the matrix model (4.1) may thereby be attributed to the $U(N)$ symmetry of the $N\times N$ matrix model of the Hubbard-Stratonovich field X. This then has the potential of presenting an exact, non-perturbative solution of the generalized noncommutative quantum field theory with action (2.1) and interaction potential given by (2.9). Furthermore, this technique has the advantage of straightforwardly supplying an explicit formula for the generating functional (2.13) of the connected Green's functions of the model.

D. Explicit solution of the master equation at large N

To solve the non-linear integral equation (4.25), we will reduce it to a standard Riemann-Hilbert problem of the type that appears for ordinary hermitean one-matrix models. For this, we introduce the resolvent function of the external field E which is defined as the Hilbert transform

$$\omega(z)=\int_{a_1}^{a_2}\mathrm{d}\lambda~\frac{\rho(\lambda)}{z-\lambda} \tag{D.1}$$

of the spectral density (4.24), and the function

$$\Omega(z)=\int_{a_1}^{a_2}\mathrm{d}\lambda~\frac{\rho(\lambda)\,W(\lambda)}{z-\lambda}\,. \tag{D.2}$$

Both of these functions are analytic everywhere in the complex z-plane, except on the support $[a_1, a_2]$ of the spectral distribution where they each have a branch cut. Let

$$\omega_\pm(\lambda) = \frac{1}{2}\Big(\omega(\lambda + \mathrm{i}\,0) \pm \omega(\lambda - \mathrm{i}\,0)\Big)\,, \qquad \lambda \in [a_1, a_2] \tag{D.3}$$

be the continuous and singular parts of the function $\omega(z)$ across its branch cut. We similarly define $\Omega_\pm(\lambda)$ for $\lambda \in [a_1, a_2]$. Note that $\omega_-(\lambda) = \mathrm{i}\,\pi\,\rho(\lambda)$ and $\Omega_-(\lambda) = \mathrm{i}\,\pi\,W(\lambda)\,\rho(\lambda)$.

From the definitions (D.1) and (D.2) it follows that the singular parts of these functions are related through

$$\Omega_-(\lambda) = W(\lambda)\,\omega_-(\lambda)\,, \qquad \forall\lambda \in \mathbb{R}\,. \tag{D.4}$$

Furthermore, from the large N Schwinger-Dyson equations (4.25) it follows that the continuous part of (D.2) obeys

$$\tilde g\,\Omega_+(\lambda) = \tilde g\,W^2(\lambda) - \Big(\lambda - \tilde g\,\omega_+(\lambda)\Big)\,W(\lambda) - 1\,, \qquad \lambda \in [a_1, a_2]\,. \tag{D.5}$$

Assuming $\tilde g \neq 0$, this suggests the analytic ansatz

$$\Omega(z) = W^2(z) - \left(\frac{z}{\tilde g} - \omega(z)\right)\,W(z) - \frac{1}{\tilde g}\,, \qquad \forall z \in \mathbb{C}\,. \tag{D.6}$$

Substituting (D.6) into (D.4) and (D.5) then implies the respective restrictions

$$W_-(\lambda)\,\left(\frac{\lambda}{\tilde g} - \omega_+(\lambda) - 2\,W_+(\lambda)\right) = 0\,, \qquad \forall\lambda \in \mathbb{R}\,, \tag{D.7}$$

$$W_-(\lambda)\,\Big(W_-(\lambda) + \omega_-(\lambda)\Big) = 0\,, \qquad \lambda \in [a_1, a_2] \tag{D.8}$$

on the real-valued analytic function $W(z)$ in the complex z-plane. In deriving (D.7) from (D.6) we have assumed that $W(z)$ has no poles.

In addition to (D.7) and (D.8), a third restriction on the function $W(z)$ is imposed by the boundary conditions required to solve the Schwinger-Dyson equations. From (4.25) it follows that there are two branches of solution, one with the asymptotic behaviour $W(z) \simeq \frac{z}{\tilde g}$ for $z \to \infty$, and the other with $W(z) \simeq -\frac{1}{z}$ for $z \to \infty$. We will take the latter boundary condition, as it is the one which matches that of the perturbative solution (4.27) of the complex matrix model. This branch reduces continuously to the gaussian solution $W^{(0)}(\lambda) = -\frac{1}{\lambda}$ at $\tilde g = 0$.[7] Thus we will in addition require

$$W(z) \simeq -\frac{1}{z} + O\left(\frac{1}{z^2}\right) \qquad \text{as} \qquad z \to \infty\,. \tag{D.9}$$

[7]The former asymptotic behaviour is the pertinent boundary condition to use in solving the corresponding hermitean matrix model in an external field, obtained by the formal substitution $X = M^\dagger M$ in the matrix integral (3.26), along with the appropriate change of integration domain. This gaussian hermitean matrix integral is proportional to $\mathrm{e}^{\,N\ \mathrm{Tr}\ E^2/2\tilde g}$, and it thereby produces a solution which is singular at $\tilde g = 0$. More precisely, the loop equation (4.25) is very similar to that of the Kontsevich-Penner model [69]–[71] whose solution is given by the singular branch. While their equations of motion are the same, the complex and hermitean matrix models differ in the choice of boundary conditions. The situation here is in marked contrast to that in the absence of the external field, whereby the difference between partition functions calculated in the large N limit with different integration domains is exponentially small [48], and hence does not affect the solution at leading order $N = \infty$.

We now solve (D.7)–(D.9) by making a one-cut ansatz, which will lead to a rational parametrization of the solution. For this, we assume that $W_-(\lambda)$ is non-vanishing on a single connected interval $[b_1, b_2]$ in the complex λ-plane. To satisfy (D.8) we must then have

$$[a_1, a_2] \cap [b_1, b_2] = \emptyset \,. \tag{D.10}$$

Since the resolvent $\omega(z)$ is analytic everywhere away from its branch cut on $[a_1, a_2]$, from (D.10) it follows that $\omega_-(\lambda) = 0$ for $\lambda \in [b_1, b_2]$. From (D.7) we then have

$$W_+(\lambda) = \frac{1}{2}\,\omega(\lambda) - \frac{\lambda}{2\tilde{g}}\,, \qquad \lambda \in [b_1, b_2]\,. \tag{D.11}$$

We can now turn (D.11) into a standard Riemann-Hilbert equation

$$\left(\frac{W(\lambda)}{\sqrt{(\lambda - b_1)(\lambda - b_2)}}\right)_- = \frac{\lambda/\tilde{g} - \omega(\lambda)}{2\,\mathrm{i}\,\sqrt{(b_2 - \lambda)(\lambda - b_1)}}\,, \qquad \lambda \in [b_1, b_2]\,. \tag{D.12}$$

It follows that the function $W(z)$ is given everywhere in the complex z-plane by the contour integral

$$W(z) = \oint_{\substack{[b_1,b_2] \\ w \neq z}} \frac{\mathrm{d}w}{4\pi\,\mathrm{i}}\,\frac{\omega(w) - w/\tilde{g}}{w - z}\,\sqrt{\frac{(z - b_1)(z - b_2)}{(w - b_1)(w - b_2)}}\,. \tag{D.13}$$

The discontinuity equation (D.12) determines the solution (D.13) uniquely up to terms which are regular at $z = 0$, and the large z behaviour (D.9) implies that the regular terms vanish. Substituting (D.1) into (D.13) and integrating along the cut $[b_1, b_2]$ then yields

$$\begin{aligned} W(z) = {} & \frac{z}{2\tilde{g}} - \frac{\sqrt{(z - b_1)(z - b_2)}}{2\tilde{g}} - \\ & - \frac{1}{2}\int_{a_1}^{a_2} \mathrm{d}\lambda\,\frac{\rho(\lambda)}{z - \lambda}\,\frac{\sqrt{(z - b_1)(z - b_2)} - \sqrt{(\lambda - b_1)(\lambda - b_2)}}{\sqrt{(\lambda - b_1)(\lambda - b_2)}}\,. \end{aligned} \tag{D.14}$$

The large z behaviour (D.9) also generates two boundary conditions which unambiguously determine the branch points b_1 and b_2 of the function $W(z)$. Expanding the right-hand side of (D.14) for $z \to \infty$ we encounter a constant term, which must vanish, and a term proportional to $1/z$, with known residue -1. This supplements (D.14) with the respective constraints

$$\begin{aligned} b_1 + b_2 &= -2\tilde{g}\int_{a_1}^{a_2} \mathrm{d}\lambda\,\frac{\rho(\lambda)}{\sqrt{(\lambda - b_1)(\lambda - b_2)}}\,, \\ 3\left(b_1^2 + b_2^2\right) + 2\,b_1 b_2 + 8\tilde{g} &= -8\tilde{g}\int_{a_1}^{a_2} \mathrm{d}\lambda\,\frac{\lambda\,\rho(\lambda)}{\sqrt{(\lambda - b_1)(\lambda - b_2)}}\,. \end{aligned} \tag{D.15}$$

E. Calculation of the convolution kernels

The trace formula (5.6) for the integration kernel $\mathcal{G}_L$ can be written in terms of the generating function (3.8) for the Landau wavefunctions as

$$\begin{aligned}\mathcal{G}_L(x_1,\dots,x_L) = \left(\frac{1}{\sqrt{4\pi\theta}}\right)^{L-2} \prod_{I=1}^{L} \int \frac{\mathrm{d}u_I\ \mathrm{d}\overline{u_I}}{\mathrm{i}\,\pi}\ \mathrm{e}^{-|u_I|^2} \times \\ \times \mathcal{P}_{\overline{u_1},u_2}(x_L)\,\mathcal{P}_{\overline{u_2},u_3}(x_{L-1})\cdots\mathcal{P}_{\overline{u_L},u_1}(x_1)\,. \end{aligned} \tag{E.1}$$

This representation can be proven by substituting the definitions (3.7) into (E.1) and repeatedly applying the integral identity

$$\int \frac{\mathrm{d}u\ \mathrm{d}\overline{u}}{\mathrm{i}\,\pi}\ \mathrm{e}^{-|u|^2}\ u^\ell\ \overline{u}^m = \ell!\ \delta_{\ell m} \tag{E.2}$$

to reduce (E.1) to (5.6). Inserting the explicit forms of the generating functions (3.8) into (E.1) then leaves a set of L coupled gaussian integrals. The result of these integrations is most efficiently presented by introducing an L dimensional vector notation involving the $L \times L$ shift matrix

$$\Gamma_L = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{E.3}$$

and the complex L-vectors

$$\boldsymbol{\xi}^\dagger = (\,\overline{\xi_1}\,,\,\dots\,,\,\overline{\xi_L}\,)\,, \qquad \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_L \end{pmatrix} \tag{E.4}$$

where

$$\xi = \sqrt{B}\,z\,, \qquad \overline{\xi} = \sqrt{B}\,\overline{z} \tag{E.5}$$

are the dimensionless rescalings of the complex coordinates (3.1) on $\mathbb{R}^2$ by the magnetic length.

After some careful inspection, it is easy to see that the integrated form can then be written as

$$\mathcal{G}_L(x_1,\dots,x_L) = 2\left(\frac{1}{2\pi\theta}\right)^{L-1}\ \det(\mathbb{1}+\Gamma_L)^{-1}\ \mathrm{e}^{\boldsymbol{\xi}^\dagger\,\Gamma_L(\mathbb{1}+\Gamma_L)^{-1}\,\boldsymbol{\xi}-\frac{1}{2}\,\boldsymbol{\xi}^\dagger\cdot\boldsymbol{\xi}}\,. \tag{E.6}$$

This same result could also have been obtained directly from (5.3) by using the Fourier representation of the delta-function. The expression (E.6) as it stands is formal, because while for L odd all eigenvalues of $\mathbb{1}+\Gamma_L$ are non-zero, for L even there is exactly one zero eigenvalue and the matrix $\mathbb{1}+\Gamma_L$ is singular. We will therefore regulate the expression

(E.6) by replacing the shift matrix Γ_L with $\alpha\,\Gamma_L$ and at the end take the limit $\alpha \to 1$. Thus we will instead compute

$$\mathcal{G}_L(x_1,\dots,x_L) = 2\left(\frac{1}{2\pi\theta}\right)^{L-1} \lim_{\alpha\to 1}\ \det(\mathbb{1}+\alpha\,\Gamma_L)^{-1}\ \exp\left(-\frac{1}{2}\,\boldsymbol{\xi}^\dagger\,\frac{\mathbb{1}-\alpha\,\Gamma_L}{\mathbb{1}+\alpha\,\Gamma_L}\,\boldsymbol{\xi}\right). \tag{E.7}$$

The determinant in (E.7) can be computed by expanding in minors along the first row to produce two triangular determinants and hence get

$$\det(\mathbb{1}+\alpha\,\Gamma_L) = \begin{vmatrix} 1 & 0 & \dots & 0 & \alpha \\ \alpha & 1 & \dots & 0 & 0 \\ 0 & \alpha & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha & 1 \end{vmatrix} = 1-(-\alpha)^L\,. \tag{E.8}$$

The calculation of the inverse matrix in (E.7) is also straightforward owing to the identity $\Gamma_L^L = \mathbb{1}$, which yields

$$(\mathbb{1}+\alpha\,\Gamma_L)^{-1} = \sum_{k=0}^{\infty}(-\alpha)^k\,\Gamma_L^k = \frac{1}{1-(-\alpha)^L}\,\sum_{p=0}^{L-1}(-\alpha)^p\,\Gamma_L^p\,. \tag{E.9}$$

Since the $p^{\rm th}$ power of the shift matrix generates a cyclic permutation of order p when acting on L-vectors, i.e.

$$(\Gamma_L^p\,\boldsymbol{\xi})_I = \xi_{(I+p)_{\mathrm{mod}\,L}}\,, \tag{E.10}$$

after a reshuffling of indices the quadratic form in (E.7) can be written as

$$\boldsymbol{\xi}^\dagger\,\frac{\mathbb{1}-\alpha\,\Gamma_L}{\mathbb{1}+\alpha\,\Gamma_L}\boldsymbol{\xi} = \frac{1}{1-(-\alpha)^L}\left[\Bigl(1+(-\alpha)^L\Bigr)\sum_{I=1}^{L}\overline{\xi_I}\,\xi_I + 2\sum_{I<J}(-\alpha)^{J-I}\Bigl(\overline{\xi_I}\,\xi_J+(-\alpha)^L\overline{\xi_J}\xi_I\Bigr)\right]. \tag{E.11}$$

To get the final result for the kernel (E.7), the cases of L odd and of L even should now be treated separately.

When $L=2r+1$ is odd, the limit $\alpha\to1$ is non-singular, and hence we can simply set $\alpha=1$ in (E.8) and (E.11) to compute (E.7) as

$$\mathcal{G}_{2r+1}(x_1,\dots,x_{2r+1}) = \left(\frac{1}{2\pi\theta}\right)^{2r}\ \exp\left(-\,\mathrm{i}\,\sum_{I<J}(-1)^{J-I}\,x_I\cdot B x_J\right). \tag{E.12}$$

When $L=2r$ is even, the regulator α must be removed carefully. For this, we set $\alpha=1-\varepsilon$, and expand (E.8) and (E.11) to second order in $\varepsilon\downarrow0$ to get

$$\det(\mathbb{1}+\alpha\,\Gamma_{2r}) = 2r\varepsilon + O\left(\varepsilon^2\right)\,, \tag{E.13}$$

$$\begin{aligned}\boldsymbol{\xi}^\dagger\,\frac{\mathbb{1}-\alpha\,\Gamma_{2r}}{\mathbb{1}+\alpha\,\Gamma_{2r}}\,\boldsymbol{\xi} &= \frac{B}{r\varepsilon}\left(\sum_{I=1}^{2r}(-1)^I\,x_I\right)^2 - B\left(\sum_{I=1}^{2r}(-1)^I\,x_I\right)^2 - \\ &\quad -\frac{2\,\mathrm{i}}{r}\,\sum_{I<J}(-1)^{J-I}\,(J-I-r)\,x_I\cdot Bx_J + O(\varepsilon)\,. \end{aligned} \tag{E.14}$$

The first term in (E.14) combines with (E.13) to produce a delta-function in the limit $\varepsilon \downarrow 0$ owing to the identity

$$\lim_{\varepsilon\downarrow 0} \frac{B}{2\pi\varepsilon}\ \mathrm{e}^{\,-\frac{B}{2\varepsilon}\,|x|^2} = \delta^{(2)}(x)\,. \tag{E.15}$$

This leads to the expression (5.7) quoted in the main text.

F. Calculation of the free propagator

In this appendix we shall compute the propagator (5.11). We will start by evaluating the Landau heat kernel

$$P_t(x,y) \equiv \langle x|\,\mathrm{e}^{\,-t\,\hat{\mathsf{D}}^2}|y\rangle = \sum_{\ell,m=1}^{\infty} \mathrm{e}^{\,-4B(\ell-\frac{1}{2})t}\ \phi_{m,\ell}(x)\,\phi_{\ell,m}(y) \tag{F.1}$$

for $0 \leq t < \infty$. For this, we will find it convenient to compute the function

$$\chi_t(k,x) = \mathrm{e}^{\,t\,\mathsf{D}^2}\ \mathrm{e}^{\,\mathrm{i}\,k\cdot x}\ \phi_0(x) = \int \mathrm{d}^2y\ P_t(x,y)\ \mathrm{e}^{\,\mathrm{i}\,k\cdot y}\ \phi_0(y)\,, \tag{F.2}$$

where ϕ_0 is the ground state wavefunction (3.6). By a simple calculation we find

$$\chi_t = \mathrm{e}^{\,t\,\mathsf{D}^2}\ \mathrm{e}^{\,\mathrm{i}\,(\mathsf{c}+\mathsf{c}^\dagger)}\ \phi_0 \tag{F.3}$$

where

$$\mathsf{c} = \overline{\kappa}\,\mathsf{a} + \kappa\,\mathsf{b}\,, \qquad \kappa = \frac{1}{\sqrt{4B}}\left(k_1 + \mathrm{i}\,k_2\right), \tag{F.4}$$

with a and b the ladder operators (3.2). By using the Baker-Campbell-Hausdorff formula with $[\mathsf{c},\mathsf{c}^\dagger] = 2|\kappa|^2$ and $\mathsf{c}\,\phi_0 = 0$ we obtain

$$\chi_t = \exp\left(\mathrm{i}\ \ \mathrm{e}^{\,t\,\mathsf{D}^2}\ \mathsf{c}^\dagger\ \mathrm{e}^{\,-t\,\mathsf{D}^2}\right)\ \mathrm{e}^{\,-|\kappa|^2}\ \mathrm{e}^{\,t\,\mathsf{D}^2}\ \phi_0\,. \tag{F.5}$$

From the oscillator representation (3.16) of the Landau hamiltonian at $B\theta = 1$ and (3.22) we get

$$\chi_t = \mathrm{e}^{\,-2Bt}\ \mathrm{e}^{\,-|\kappa|^2}\ \exp\,\mathrm{i}\,\left(\kappa\ \mathrm{e}^{\,-4Bt}\ \mathsf{a}^\dagger + \overline{\kappa}\,\mathsf{b}^\dagger\right)\ \phi_0\,. \tag{F.6}$$

Inserting $\phi_0 = \exp\,\mathrm{i}\,(\kappa\ \mathrm{e}^{\,-4Bt}\ \mathsf{b} + \overline{\kappa}\,\mathsf{a})\ \phi_0$ into (F.6) and using the Baker-Campbell-Hausdorff formula again yields

$$\chi_t(k,x) = \mathrm{e}^{\,-2Bt}\ \mathrm{e}^{\,-|\kappa|^2}\ \mathrm{e}^{\,|\kappa|^2\exp(-4Bt)}\ \exp\,\mathrm{i}\,\left(\kappa\ \mathrm{e}^{\,-4Bt}\ \overline{\xi} + \overline{\kappa}\,\xi\right)\ \phi_0(x)\,, \tag{F.7}$$

where we have used (3.2) and the rescaling (E.5).

Along with (F.2), the expression (F.7) allows us to compute the heat kernel (F.1) via the Fourier transform

$$\begin{aligned} P_t(x,y) = \frac{\phi_0(x)}{\phi_0(y)} \int \frac{\mathrm{d}^2k}{(2\pi)^2}\ \mathrm{e}^{\,-\mathrm{i}\,k\cdot y}\ \mathrm{e}^{\,-2Bt}\ \mathrm{e}^{\,-|\kappa|^2\left(1-\exp(-4Bt)\right)} \times \\ \times \exp\,\mathrm{i}\,\left(\kappa\ \mathrm{e}^{\,-4Bt}\ \overline{\xi} + \overline{\kappa}\,\xi\right)\,. \end{aligned} \tag{F.8}$$

The integral in (F.8) is gaussian , and after a straightforward computation we arrive at the final result

$$P_t(x,y) = \frac{B}{2\pi\sinh(2Bt)}\ \mathrm{e}^{\,-\frac{B}{2}\coth(2Bt)\,|x-y|^2}\ \mathrm{e}^{\,-\,\mathrm{i}\,x\cdot By}\,. \tag{F.9}$$

Note that from (E.15) it follows that $P_t(x,y)\to\delta^{(2)}(x-y)$ for $B\downarrow 0$, as it should. Finally, the propagator (5.11) is given by

$$\begin{aligned} C_\mu(x,y) &= \int_0^\infty \mathrm{d}t\ \mathrm{e}^{\,-t\mu^2}\,\langle x|\,\mathrm{e}^{\,-t\,\hat{\mathsf{D}}^2}|y\rangle \\ &= \frac{B}{2\pi}\ \mathrm{e}^{\,-\,\mathrm{i}\,x\cdot By}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{\,-t\mu^2}\ \frac{\mathrm{e}^{\,-\frac{B}{2}\coth(2Bt)\,|x-y|^2}}{\sinh(2Bt)}\,. \end{aligned} \tag{F.10}$$

The change of variables

$$u = \frac{B\,|x-y|^2}{2}\Big(\coth(2Bt)-1\Big) \tag{F.11}$$

then brings the propagator (F.10) into the form (5.12) given in the main text.

G. Scaling limit of the function $\gamma_\ell(x,y)$

To compute the sum over Landau eigenfunctions (5.18), it is convenient to introduce the generating function

$$\gamma(x,y;t) = \sum_{\ell=1}^\infty \frac{t^{2(\ell-1)}}{(\ell-1)!}\ \gamma_\ell(x,y)\,. \tag{G.1}$$

Proceeding as with the representation (E.1), by using (5.18) we may then express (G.1) in terms of the generating functions (3.7) as

$$\gamma(x,y;t) = 4\pi\theta\int\frac{\mathrm{d}u\ \mathrm{d}\overline{u}}{\mathrm{i}\,\pi}\ \mathrm{e}^{\,-|u|^2}\int_0^{2\pi}\frac{\mathrm{d}\tau}{2\pi}\ \mathcal{P}_{\overline{u}\,,\,t\ \mathrm{e}^{\,-\,\mathrm{i}\,\tau}}(x)\,\mathcal{P}_{t\ \mathrm{e}^{\,\mathrm{i}\,\tau}\,,\,u}(y)\,. \tag{G.2}$$

Upon substituting in the explicit form (3.8), we observe that the integral over u is gaussian , and that the resulting τ integration yields the Bessel function J_0 of the first kind of order 0, giving

$$\gamma(x,y;t) = 4\ \mathrm{e}^{\,-|x-y|^2/2\theta+\mathrm{i}\,x\cdot By+t^2}\ J_0\left(\frac{2t\,|x-y|}{\sqrt{\theta}}\right). \tag{G.3}$$

By substituting the power series expansions of the exponential and Bessel functions into (G.3) and comparing with (G.1) we find the functions (5.18) for finite ℓ and θ in the form

$$\gamma_\ell(x,y) = 4(\ell-1)!\ \mathrm{e}^{\,-|x-y|^2/2\theta+\mathrm{i}\,x\cdot By}\ \sum_{k=0}^{\ell-1}\frac{(-1)^k}{(\ell-k-1)!\,(k!)^2}\ \left(\frac{|x-y|^2}{\theta}\right)^k\,. \tag{G.4}$$

We are interested in the function (G.4) in the scaling limit $\theta\to\infty$, $\ell\to\infty$ with $\frac{\ell}{\theta}$ fixed. In this limit, the exponential factor in (G.3) reduces to $\mathrm{e}^{\,t^2}$. The extraction of large orders in ℓ in the Taylor series (G.1) is accomplished by the standard method of contour integration to write

$$\gamma_\ell(x,y) = (\ell-1)! \oint_{t=0} \frac{\mathrm{d}t}{2\pi\,\mathrm{i}\,t^{2\ell-1}}\ \gamma(x,y;t)\,. \tag{G.5}$$

By rescaling the integrand of (G.5) as $t = \sqrt{\ell-1}\,w$ and using (G.3) we get

$$\gamma_\ell(x,y) = 4\,(\ell-1)!\ \mathrm{e}^{\,\ell-1} \oint_{w=0} \frac{\mathrm{d}w}{2\pi\,\mathrm{i}\,w}\ J_0\left(2w\,|x-y|\,\sqrt{\frac{(\ell-1)}{\theta}}\right)\ \mathrm{e}^{\,(\ell-1)\,(w^2-2\ln w)}\,. \tag{G.6}$$

In the large ℓ limit, the integral (G.6) can be evaluated by using the saddle-point approximation. There are two saddle points, at $w = \pm 1$. By expanding (G.6) near these saddle points and using the Stirling approximation $\ell! \simeq \sqrt{2\pi\ell}\ \mathrm{e}^{\,-\ell}\,\ell^{\ell}$ for $\ell \to \infty$, we finally find

$$\gamma_\ell(x,y) = 4\,J_0\left(2\,|x-y|\,\sqrt{\frac{\ell}{\theta}}\right)\,, \tag{G.7}$$

which on trading the integers ℓ for the eigenvalues λ_ℓ defined in (4.31) yields the result (5.19) in the main text.

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